CALCULUS IS ALGEBRA

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1. Introduction. Presumably everyone who teaches calculus will agree that some knowledge of algebra is a prerequisite for a course in calculus. However, there can always be an argument as to how much is enough. The basic fact to be assimilated is that the real numbers form a complete ordered field, but this fact can be learned operationally without being assimilated conceptually (or at least without being totally and explicitly assimilated on the conceptual level). In any case, the basic problem is pedagogical and not mathematical and involves determining the level of algebraic conceptualization which optimizes the learning of calculus.

The main point of the present paper is that the use of a few extremely elementary notions and facts of ring theory enables one to do calculus in the hyperreal number system, thus reducing calculus totally to algebraic manipulation. In other words, with only a slight increase in conceptual sophistication, there is a substantial payoff in the form of reduced operational complexity. Since reducing operational complexity through conceptualization is the very heart of mathematical activity, I feel that this approach to the teaching of calculus lays serious claim to being the "correct one."

2. Algebraic prerequisites. To construct and use the hyperreal numbers in their simplest form, we need only the following notions and facts.

Notions.
(1) Ring with unit.
(2) Subring and ideal.
(3) Quotient ring by an ideal.
(4) Integral domain and field.
(5) Prime ideal and maximal ideal.

Facts.
(1) A maximal ideal is prime.
(2) Any proper ideal is contained in a maximal one.
(3) Every field is an integral domain.
(4) A quotient ring is a field iff the dividing ideal is maximal.

We will now see how these facts and notions are used to construct and then to use the hyperreal numbers. We take the real numbers R as given and understood on whatever level of conceptual sophistication is deemed appropriate by the instructor.

3. The construction of R^* from R. We begin by forming the ring R^N of all sequences of real numbers (N is the set of positive integers). Addition and multiplication of sequences is done in the obvious component-by-component fashion. The zero sequence is the zero of the ring and the sequence with constant value 1 is the unit of the ring R^N. In fact, R is embedded into R^N by the diagonal correspondence $r \mapsto \vec{r}$ where $\vec{r}$ is the sequence which is constantly r, $\vec{r}_n = r$ for all

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$n \in N$.

R^N is not a field, however, since it is not even an integral domain. For example, consider the two sequences s and t where s is 1 on the evens and 0 on the odds while t is 1 on the odds and 0 on the evens. Then $st = 0$, but neither s nor t is 0.

We want now to obtain a quotient ring R^N/M which is a field. We are thus looking for an appropriate maximal ideal M in R^N. We want the resulting field, called R^*, to satisfy (at least) the following conditions: (I) It must be totally ordered and must contain R as an ordered subfield (it must in fact preserve the embedding of R into R^N). (II) It must be non-Archimedean over R, i.e., it must contain elements ω which are bigger than every real number. (III) Every finitary operation $f: R^n \to R$, $n \geqslant 1$, and every finitary relation $K \subset R^n$, $n \geqslant 1$, must extend canonically to R^*. In each case, the extension defined on R^* must be identical on R with the original function or relation defined on the field R, and of the same arity. (IV) Moreover, for each $n \geqslant 1$, complements, unions, and characteristic functions of n-ary relations are preserved under canonical extensions. The composition of operations is preserved under the canonical extension of operations, and, for $n \geqslant 1$, the extension of an n-ary projection on R is the corresponding projection on R^*, and the extension of a constant function is constant. The field operations on R^* are the canonical extensions of the corresponding field operations on R.†

We now proceed to find an appropriate maximal ideal M.

DEFINITION 1. Let a be any sequence in R^N. By the *support* of a we mean the set $\sigma(a) \subset N$ of indices n such that $a_n \neq 0$.

Since the ideal becomes the zero element in the quotient ring obtained by dividing by the ideal in question, we are looking for an ideal of sequences having lots of zero values.

DEFINITION 2. F is the set of sequences a having a finite support set $\sigma(a)$.

Thus, the sequences in F are sequences which are zero everywhere except on some finite set of indices.

PROPOSITION 1. *F is a proper ideal in R^N.*

Proof. A simple, straightforward verification that the ideal properties are true of F.

Since F is a proper ideal, it is contained in a maximal ideal M.

DEFINITION 3. $R^* = R^N/M$. Elements of R^* are called *hyperreal numbers.*‡

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†The conjunction of conditions I–IV is equivalent to the axioms of Keisler [7]. The main difference is that we here deal with n-ary relations and n-ary operations whereas Keisler deals rather with n-ary partial functions. Also, our approach here is totally "objective" in that we deal with functions and relations in extension in the way which is customary to the working mathematician, thereby avoiding any explicit appeal to logic or to such logical, syntactical notions as that of a formula. Other recent versions of nonstandard analysis (see [6]) involve far more logical notions than does Keisler and are therefore at the opposite end of the "objective-subjective" spectrum from the approach of this paper. The purpose of conditions like I–IV or Keisler's axioms is to obtain the so-called "transfer principle" which enables us to transfer wholesale from R to R^* all elementary statements (roughly, statements which quantify only over elements and not over subsets of R). This makes R^* an *elementary extension* of R (see Chang and Keisler [1] and Keisler [8]). There are some intentional redundancies in our formulation of conditions I-IV.

‡In the presence of the continuum hypothesis, the field $R^* = R^N/M$ is uniquely determined up to an isomorphism of ordered fields even though the maximal ideal $M \supset F$ is not necessarily unique (see [3]). It would therefore seem reasonable to tell students that R^* is uniquely defined for all practical purposes since models of set theory in which the axiom of choice (which we have assumed in the form of Krull's Lemma as fact (2) above) holds but CH fails are rarely, if ever, used in everyday mathematics. Also note that $R^* = R^N/M$ can be defined from R^N using ultrafilters on the powerset of the natural numbers N (see Keisler [8]). Indeed, the set S of Definition 4 and Proposition 2 of this paper is such an ultrafilter. This conversion between maximal ideals and ultrafilters is a special case of a natural bijection between filters on the powerset of N and ideals in R^N (see [2]). According to this bijection, the ideal F of Definition 2 above corresponds to the Fréchet filter, making ideals $M \supset F$ correspond to free ultrafilters. Thus, the R^* of Definition 3 above is an ultrapower of the reals R and a proper elementary extension of R. In view of the fact that the Boolean Prime Ideal theorem is somewhat weaker than the axiom of choice (see [4]), the construction using ultrafilters can be carried on under somewhat weaker ontological assumptions if this is considered desirable.

A hyperreal number is thus an equivalence class of sequences of real numbers. We use brackets [ ] to denote equivalence classes. Two sequences a and b are equivalent iff their difference $a - b \in M$. In particular, since $F \subset M$, two sequences are equivalent whenever they differ only on some finite set of indices. Also, every sequence $a \in M$ must have at least one zero value, $a_k = 0$ for some $k \in N$, for otherwise the sequence $1/a_n$ exists and the product $(1/a) \cdot a = \vec{1} \in M \Rightarrow M = R^N$, contradicting the maximality of the ideal M.

Since M is maximal, R^* is a field. Moreover, R is contained as a subfield of R^* by the obvious representation $r \mapsto [\vec{r}]$. The only thing to check is that this correspondence is injective, i.e., $[\vec{r}] = [\vec{s}] \Rightarrow r = s$. We have: $[\vec{r}] = [\vec{s}] \Leftrightarrow \vec{r} - \vec{s} = \overrightarrow{r - s} \in M$. Since the constant sequence $\overrightarrow{r - s}$ is either everywhere zero or nowhere zero, it must be constantly zero if it is in M. Thus $r = s$ if $[\vec{r}] = [\vec{s}]$.

It follows from the fact that $R \subset R^*$ that the cardinalities of R and R^* are the same (namely $2^{\aleph_0}$). Indeed, by definition the cardinality of R^* is less than or equal that of R while the above shows that the cardinality of R is less than or equal that of R^*.

We want now to extend the order relation $<$ on R to R^*. To do this smoothly, the following definition is useful.

DEFINITION 4. $X \in S \Leftrightarrow X = \sigma(a)$ for some $a \notin M$. S is the collection of all support sets of sequences not in M.

PROPOSITION 2. *The collection S satisfies the following properties*: (1) $N \in S$. (2) $X, Y \in S \Rightarrow X \cap Y \in S$. (3) $Y \supset X \in S \Rightarrow Y \in S$. (4) $\varnothing \notin S$. (5) *All cofinite sets* (*sets whose complements in N are finite*) *are in S.* (6) *For all $X \subset N$, either $X \in S$ or $(N - X) \in S$ and not both.* (7) *If $(X \cup Y) \in S$, then either $X \in S$ or $Y \in S$.* (8) *If $X \in S$, then X cannot be the support set of any sequence $a \in M$.*

Proof. (1) $N = \sigma(\vec{1})$ and $\vec{1} \notin M$. (2) If $X = \sigma(a)$ and $Y = \sigma(b)$ where $a, b \notin M$, then $X \cap Y = \sigma(a \cdot b)$. If $a \cdot b \in M$, then either $a \in M$ or $b \in M$ since M is maximal and therefore prime. Thus, $a \cdot b \notin M$ and $X \cap Y \in S$. (3) Suppose $Y \supset X \in S$, $X = \sigma(a)$ where $a \notin M$. Define the sequence b as follows:

$$b_n = \begin{cases} 1 & \text{if } n \in Y \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, $Y = \sigma(b)$ and $a \cdot b = a$. By the absorbing property of the ideal M, $a = a \cdot b \in M$ if $b \in M$. Since $a \notin M$, $b \notin M$. Thus, $Y \in S$. (4) The null set $\varnothing$ is the support set of only one sequence, namely the constantly zero sequence $\vec{0}$. Since $\vec{0} \in M$, $\varnothing \notin S$. (5) Since $F \subset M$, all sequences with finite support are in M. If some sequence a with cofinite support is in M, then let b be any sequence with finite support $\sigma(b) = N - \sigma(a)$. The sequence $(a + b) \in M$ since M is closed under addition. But the sequence $a + b$ has no zero value (its support is N) and this is impossible as we have already seen. Thus, no sequence with cofinite support is in M and hence every cofinite set is in S. (6) Consider any $X \subset N$. Let the sequence a be the characteristic function of X and let a' be the characteristic function of $N - X$. Clearly, $X = \sigma(a)$ and $N - X = \sigma(a')$. If both $a \in M$ and $a' \in M$, then $(a + a') \in M$. But the sequence $a + a'$ has no zero value which is impossible for any sequence in M. Thus, either $a \notin M$ or $a' \notin M$ and consequently either $X \in S$ or $(N - X) \in S$ by definition. Finally, if both X and $N - X$ are in S, then by (2) above their intersection $\varnothing \in S$ contradicting (4). (7) Suppose that $(X \cup Y) \in S$ but $X \notin S$ and $Y \notin S$. Then, by the property (6) of S, $(N - X) \in S$ and $(N - Y) \in S$ whence their intersection is in X by property (2). By DeMorgan's law, this intersection is the complement in N of $X \cup Y$. We thus have $(N - (X \cup Y)) \in S$ and $(X \cup Y) \in S$ contradicting property (6). Thus, either X or Y must be in S. (8) Suppose $X = \sigma(a)$, $X = \sigma(b)$, $a \in M$, and $b \notin M$. Clearly $a \neq \vec{0}$ since otherwise $X = \varnothing \in S$, contradicting property (4) above. Define a new sequence c by the rule

$$c_n = \begin{cases} 1/a_n & \text{for } n \in X \\ 0 & \text{otherwise} \end{cases}.$$

By the absorbing property of the ideal M, the product sequence $c \cdot a \in M$. The sequence $c \cdot a$ has the value 1 for every $n \in X$ and 0 elsewhere. Thus, the product $b \cdot (c \cdot a) = b$. But, again by the absorbing property of the ideal M, the product $b = b \cdot (c \cdot a) \in M$, contradicting the hypothesis that $b \notin M$. Thus, if $X \in S$ (i.e., if X is the support set of some sequence $b \notin M$), then X cannot be the support set of any sequence $a \in M$.

The usefulness of the collection S and its properties will be immediately clear from the ensuing development.

LEMMA 1. *For any* $a, b \in R^N$, $[a] = [b]$ *iff* $\{n \mid a_n = b_n\} \in S$.

Proof. $[a] = [b] \Leftrightarrow (a - b) \in M$. Now, $\{n \mid a_n \neq b_n\} = \{n \mid (a - b)_n \neq 0\} = \sigma(a - b)$. Thus, if $(a - b) \notin M$, then $\{n \mid a_n \neq b_n\} \in S$ and therefore $\{n \mid a_n = b_n\} \notin S$ by property (6) of S. This establishes the implication $\{n \mid a_n = b_n\} \in S \Rightarrow [a] = [b]$.

Going the other way, suppose $(a - b) \in M$ but $\{n \mid a_n = b_n\} \notin S$. Then, again by the property (6) of S, $S \ni \{n \mid a_n \neq b_n\} = \sigma(a - b)$ which contradicts property (8) of S. Thus, $[a] = [b] \Rightarrow \{n \mid a_n = b_n\} \in S$.

Lemma 1 gives us a useful necessary and sufficient condition for two sequences to be equivalent.

DEFINITION 5. $[a] < [b]$ iff $\{n \mid a_n < b_n\} \in S$, for $a, b \in R^N$.

By Lemma 1 and the properties of S, the relation $<$ is well-defined on equivalence classes of sequences.

THEOREM 1. *The relation* $<$ *is a total order on* R^* *which extends the usual total ordering on* R.

Proof. $\{n \mid a_n < a_n\} = \varnothing \notin S$. Thus $[a] \nless [a]$ for all $[a] \in R^*$. If $[a] < [b] < [c]$, then $\{n \mid a_n < c_n\} \supset \{n \mid a_n < b_n\} \cap \{n \mid b_n < c_n\}$ and this latter intersection is in S. Thus, by property (3) of S, the superset $\{n \mid a_n < c_n\}$ is in S and $[a] < [c]$ by definition.

These two facts establish that $<$ is a partial ordering of R^*. We now show that any two elements $[a]$ and $[b]$ of R^* are comparable. Suppose $[a] \nless [b]$, i.e., $\{n \mid a_n < b_n\} \notin S$. By property (6) of S, its complement $\{n \mid a_n \nless b_n\} = \{n \mid a_n \geqslant b_n\} \in S$. By definition, this last set $\{n \mid a_n \geqslant b_n\} = \{n \mid a_n > b_n\} \cup \{n \mid a_n = b_n\}$. We now have two sets whose union is in S. Thus, by property (7) of S, either $\{n \mid a_n > b_n\} \in S$ or $\{n \mid a_n = b_n\} \in S$, that is, by Definition 5 and Lemma 1, either $[a] > [b]$ or $[a] = [b]$.

It is not difficult to verify that $<$ is compatible with the field operations, making R^* an ordered field.

Finally, we observe that for diagonal sequences $\vec{r}$ and $\vec{s}$, $\{n \mid \vec{r}_n < \vec{s}_n\}$ is either the whole set N or else $\varnothing$ depending on whether $r < s$ holds or not. Since $N \in S$ and $\varnothing \notin S$, $[\vec{r}] < [\vec{s}] \Leftrightarrow r < s$. In other words the ordering $<$ on R^* extends the usual ordering on R. This completes the proof.

Theorem 1 shows that R^* satisfies condition I outlined above. We now turn our attention to the task of establishing property II, i.e., that R^* is non-Archimedean over R. We begin by establishing some useful terminology.

The absolute value function $|\ |: R^* \to (R^*)^+$ is defined in R^* in the usual way for ordered fields: For $a \in R^*$, $|a| = a$ if $a \geqslant 0$ and $-a$ otherwise. We also extend the ordering relation to sets of hyperreals in the usual way. This facilitates the following definition.

DEFINITION 6. A hyperreal number a such that $|a| > R^+$ is an *infinite* number. A hyperreal number a such that $|a| < R^+$ is an *infinitesimal* number. Finally, a *finite* number is a noninfinite one.

Clearly, 0 is the only real infinitesimal number. Moreover, the positive infinitesimal and the positive infinite numbers are in obvious 1–1 correspondence via inversion: $a > R^+$ if and only if $0 < 1/a < R^+$. This recalls the bijection between the real intervals $(0,1)$ and $(1,\infty)$. A finite number is one whose absolute value is smaller than some positive real integer.

We now prove that there are infinite and therefore infinitesimal hyperreal numbers.

THEOREM 2. *There exists an element* $\omega \in R^*$ *which is greater than every real number* $[\vec{r}] \in R^*$, $r \in R$.

Proof. Let $s\colon N \to R$ be the identity sequence on N, i.e., $s_n = n$ for all $n \in N$. Let $\omega = [s]$. Let any real number $[\vec{r}]$ be given. Since R is Archimedean, there exists some natural number k such that $r < k$. Thus, $\{n \mid \vec{r}_n < s_n\} \supset \{n \mid n > k\}$. But this last set is in S since it is cofinite (it is the complement of the finite set $\{1,2,\ldots,k\}$). Thus, by property (3) of S, $\{n \mid \vec{r}_n < s_n\} \in S$ and $[\vec{r}] < \omega$. But $[\vec{r}]$ was arbitrary. Thus, ω is greater than every real number: $R^+ < \omega$.

The hyperreal number $1/\omega$ will be a positive infinitesimal number as we have already seen in the informal remarks preceding Theorem 2. Thus, the existence of infinite and of infinitesimal numbers is clearly established.

Notice that $r + \omega$ will also be infinite for any real r. In particular, for r negative this can be seen as follows: If $x = \omega - r$ were finite where $r \in R^+$, then $\omega = x + r$ is the sum of two finite numbers. But the finite numbers are easily seen to be closed under addition. Thus, $x = \omega - r$ has to be infinite. Hence the order structure of the positive infinite numbers contains at least the set $R + \omega$ which is order-isomorphic to the reals. In fact, the order structure of the positive infinite numbers is considerably more complicated as will be clear once the structure of the ring of finite numbers is examined further on.†

We conclude the present section by establishing III, namely that every function and relation on R has a canonical extension to R^*, and IV.

Given a real function $f\colon R \to R$, we define its canonical extension to R^* in the following manner: First, we extend f to R^N by the obvious rule: $f(s)_n = f(s_n)$ where $s\colon N \to R$ and $n \in N$. Finally, we extend f to equivalence classes $[s]$ of sequences by the definition $f([s]) = [f(s)]$. We must, however, justify that this is well-defined, i.e., that $[f(s)] = [f(t)]$ whenever we have $[s] = [t]$. By Lemma 1, this last condition is equivalent to the condition $\{n \mid s_n = t_n\} \in S$. Since $f(s_n) = f(t_n)$ whenever $s_n = t_n$, $\{n \mid f(s)_n = f(t)_n\} \supset \{n \mid s_n = t_n\}$ and so $\{n \mid f(s)_n = f(t_n)\}$ is in S by (3) of Proposition 2. Thus $[f(s)] = [f(t)]$ by Lemma 1. By the way we have extended functions f from domain R to domain R^N, it is clear that the extension of f will yield a diagonal sequence $\overrightarrow{f(r)} = f(\vec{r})$ when applied to a diagonal sequence $\vec{r}$. Since it is equivalence classes $[\vec{r}]$ that represent real numbers r as elements of R^*, the extension of f to R^* will yield the same corresponding values when applied to $[\vec{r}]$ as it did when applied to r to begin with, i.e., $f([\vec{r}]) = [\overrightarrow{f(r)}]$. Generalization to the case of a function $f\colon R^n \to R$, $n \in N$, of any nonzero finite number of variables is immediate. This establishes condition III for functions.

Condition IV is also immediate for functions since extension clearly preserves functional composition, projections, constant functions, and the field operations. We give the same names to functions on R and their canonical extensions to R^*.

Let us now examine the case of relations on R. The extension to R^* of a binary relation K on R is defined by: $[s]K[t]$ if and only if $\{n \mid s_n K t_n\} \in S$. This is well-defined on equivalence classes of sequences by Lemma 1 and the properties of the set S. Since $N \in S$ and $\varnothing \notin S$, it follows that K will hold between two equivalence classes of diagonal sequences, $[\vec{r}]$ and $[\vec{s}]$, when and only when K holds between r and s, i.e., rKs. Thus, the extension of K to R^* will continue to hold for exactly the same couples $\langle[\vec{r}],[\vec{s}]\rangle$ for which rKs on R. Since $N \in S$, the extension of $R \times R$ to R^* will be the universal relation $R^* \times R^*$. In fact, extension preserves complements, unions, and characteristic functions of binary relations, by the properties of S and the relevant definitions. We give the same names to relations on R and to their canonical extensions on R^*. Finally, the generalization to the case of n-ary relations, for $n \geqslant 1$, is immediate. This establishes conditions III and IV for relations and completes our task of constructing a field satisfying conditions I–IV.‡

Notice that the $<$ relation that we have previously defined on R^* (Definition 5) is the canonical extension of $<$ on R. Thus, the absolute value function $|\ |$ we have defined on R^* is the canonical extension of $|\ |$ defined on R. Also, the set N of natural numbers has a canonical

†See [5] for a study of the order structure of R^*.

‡One mildly subtle point remains to be checked: n-ary operations f are also $(n+1)$-ary functional relations. We need to know that the extension of such an f as a function is the same as the extension of f as a relation, in all cases. It is, and the verification is left to the reader.

extension $N^* \subset R^*$: $[s] \in N^* \Leftrightarrow \{n \mid s_n \in N\} \in S$. Elements of $N^* - N$ are infinite elements of R^* and are called *infinite natural numbers*.

The practical meaning of conditions III and IV is that all those identities and relationships which hold on R will continue to hold for the corresponding extensions defined on R^*. For example, the identity $\text{Sin}^2 x + \text{Cos}^2 x = 1$ for all real numbers x will hold, for all hyperreal numbers z, for the canonical extensions of Sin and Cos. The extension to R^* of the function f defined on R by the rule $f(x) = x^2$ will be defined by the rule of squaring when applied to any hyperreal number z. Etc.

Having now constructed R^*, we turn to a study of its structure.

4. The structure of R^*. An interesting aspect of the approach to the calculus via the hyperreal numbers is that we will henceforth have only occasional need for our construction of R^*. Now that we know that R^* satisfies the conjunction of I, II, III, and IV, we can "throw away" our construction of it. Of course, it is both pedagogically and theoretically useful to have a particular model of R^* at hand to help us think about the structure of the hyperreals.

This situation is analogous to the study of the real numbers in which a model of a complete ordered field can be constructed in several different ways, e.g., equivalence classes of Cauchy sequences or Dedekind cuts. The analogy is not perfect, however, since all models of complete ordered fields are isomorphic to each other, whereas there are nonisomorphic structures satisfying the conjunction of I–IV. Nevertheless, the analogy is close enough to permit a purely axiomatic treatment of calculus in the hyperreals as Keisler does in [7].

We let R_0 be the set of all finite hyperreal numbers. We have already noted in passing that

THEOREM 3. *R_0 is a subring of R^* but not a subfield.*

Proof. It is easy to check that the sum, difference, and product of finite numbers are finite. Infinitesimals are finite, and the reciprocals of nonzero infinitesimals are infinite, contradicting closure under division.

Let I denote the set of infinitesimal numbers. It is easy to check that I is an ideal in R_0. Thus, in particular, $I \cdot R_0 \subset I$. The equivalence classes $x + I$, $x \in R_0$, determined by I are called *monads*. Furthermore,

THEOREM 4. *I is a maximal ideal of R_0.*

Proof. Any ideal J that properly contains I must contain at least one finite, noninfinitesimal number x. But $1/x$ is also finite, noninfinitesimal and so $1 = (1/x) \cdot x \in J$ which yields $J = R_0$, i.e., J is not a proper ideal of R_0. This establishes the maximality of I.

As in our earlier construction, we again have a field, R_0/I. We establish that, in fact, $R \cong R_0/I$.

Let g: $R_0 \to R_0/I$ be the canonical homomorphism of R_0 onto the quotient field R_0/I. For all $x \in R_0$, $g(x) = x + I$. We show that there is exactly one real number r in each equivalence class $g(x) = x + I$. Uniqueness is easy, for if r_1 and r_2 are real numbers such that $r_1 - r_2 \in I$, then $r_1 - r_2 = 0$ since the difference of two real numbers is real and 0 is the only real number in the ideal I of infinitesimals.

Let, now, $x + I$ be some equivalence class for $x \in R_0$. We assume $x > 0$, the case $x \leqslant 0$ being handled symmetrically. If x is real, there is nothing to prove. Otherwise, let $X = \{r \in R \mid r \leqslant x\}$. The set X of real numbers is not empty since $0 \in X$. Furthermore, X is bounded above since x is finite and positive (in other words there are real numbers t such that $t > x \geqslant X$). X is thus a nonempty set of real numbers bounded above. By the completeness property of the field of real numbers R, X has a real supremum $s \geqslant X$. If $s > x$, then $s \in x + I$, for otherwise $|s - x| = s - x > r$ for some positive real r, which immediately gives $s > s - r > x \geqslant X$, and $s - r$ is a real bound of X smaller than s, a contradiction. On the other hand, if $s < x$, then for all positive real numbers r, $r + s > s \geqslant X$ which means that $r + s \notin X$. By the definition of X, then, $r + s > x$ for every positive real r. In other words, $r > x - s = |s - x|$ for every positive real r. Hence, by

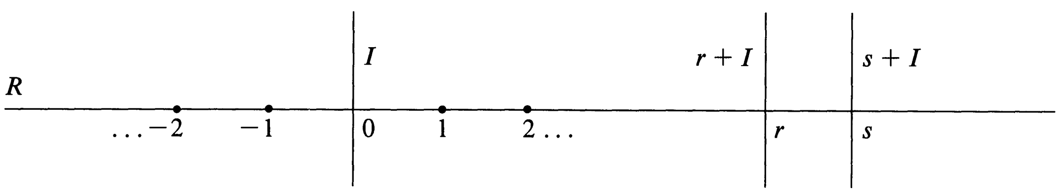
Definition 6, $s - x$ is infinitesimal which means $s \in x + I$. Thus, every monad $x + I$ is of the form $r + I$ for some unique real number r. We have proved:

THEOREM 5. *Every finite hyperreal number x is of the form $r + i$ where r is real and i is infinitesimal.*

We call r the *real part* and i the *infinitesimal part* of x. We define a mapping rp: $R_0 \to R$ which associates the real part of x to each finite hyperreal number x ("rp" stands for "real part"). The mapping rp is the composite of g: $R_0 \to R_0/I$ and the isomorphism between R_0/I and R and is therefore a ring homomorphism of R_0 onto R.

Notice that, for r, s real and i, j infinitesimal, $r + i < s + j$ if and only if $r < s$ or $r = s$ and $i < j$. We can therefore consider that the finite hyperreal numbers are ordered pairs of the form $\langle r, i \rangle$, r real and i infinitesimal, which are ordered lexicographically.

Let us now recall that R^*, and hence I, have the same cardinality as the real line R. We can thus use some fixed bijection between I and R to establish a bijection β between the (linearly) ordered ring R_0 and the real Cartesian plane $R \times R$. It is understood that this bijection is the identity on R (in other words, the abscissa is invariant under β^{-1}). We can use the bijection β: $R_0 \to R^2$ to induce on R^2 an ordered ring structure, making R^2 isomorphic to R_0 as an ordered ring. This yields the following geometric model of R_0:



Here, the abscissa is the usual real line and the ordinate is the ideal I of infinitesimals. At each point r of the real line the unique perpendicular at r represents the set $r + I$ which is the r-translation of the ideal I. It is the monad determined by r. Every finite hyperreal number is an ordered pair $\langle r, i \rangle$ whose real part is obtained by projecting onto the real axis and whose infinitesimal part is obtained by projecting onto the infinitesimal axis (the ordinate).

R_0 is an ordered ring, and two monads $r + I$ and $s + I$ are ordered as are their real parts. Geometrically, this order is given as increasing from left to right along the real line. In particular, the monad I of 0 is less than every positive real number.

It would be nice to be able to think of the order of I as given by the rule "greater means higher up," but the order-type of the infinitesimals does not permit this.† Even though the set I is bijective with the points on the real line, its order-type has no countable basis and cannot therefore be legitimately thought of as a sub-type of the order-type of the reals. Nevertheless, I is totally ordered and the translation operation $r + I$ preserves this order, so that each monad is ordered in the same way.

The positive infinite numbers can be thought of as "beyond" the real line in the positive direction while the negative infinites are beyond the real line in the negative direction. The positive infinite numbers are in 1-1 correspondence with the positive infinitesimals via inversion. A similar relationship holds for the negative infinite numbers.

Returning one last time to our geometrical model of R_0, let us imagine that we "compress" each vertical line $r + I$ into the point r. We now have simply the usual, one-dimensional geometrical model of the real line, except that now we think of the points on the real line as having a structure: each point on the abscissa *is* the monad $r + I$, which can be represented (as

†Except, of course, by analogy. Those who consider that the whole "real numbers = Euclidean line" concept is an analogy in the first place will not be restrained here. In any case, we can suppose a bijection between the positive infinitesimals and the portion of I above the real line, and this separation is preserved under translation.

usual) by the unique real number r it contains.† Since points are so small, their inner structure can be detected only by the "infinitesimal microscope" of [**7**].

The geometric model of R_0, both with and without the above modification, can be pedagogically useful in visualizing the structure of the hyperreals and in thinking about functions defined on the reals and their extensions to the hyperreals.

5. Using R^*. At this point, the development of the calculus using R^* is immediately accessible, without further theoretical development. A real function f is differentiable at a real point r iff, for any two nonzero infinitesimals i and j, the ratios $[f(r+i)-f(r)]/i$ and $[f(r+j)-f(r)]/j$ are in the same monad. In that case,

$$f'(r) = \mathrm{rp}\left(\frac{f(r+i)-f(r)}{i}\right)$$

where i is any nonzero infinitesimal.

The derivatives of polynomial functions can be simply calculated in short order and the usual differentiation formulas directly derived. The usual geometrical interpretations can be given and applications to maxima, minima, extrema, etc. speedily developed. Even the notion of a continuous function is not necessary to these developments, though it is easily accessible as well.

In these, and all subsequent developments, there is not only a practical simplification, but a theoretical one: the usual contravariant ε-δ limit definition is replaced by the equivalent covariant one involving infinitesimals. We say that $\mathrm{Lim}_{x\to a} f(x) = L$ if and only if $f(x) - L \in I$ whenever $x - a$ is, $x \neq a$. In particular, a continuous real function f is precisely one which preserves monads: $f\colon R \to R$ is continuous if, for every $r \in R$, $i \in I$, $f(r+i) = f(r) + j$ where $j \in I$. This is easily motivated and understood in terms of the geometrical model of R_0 above.

There is not only conceptual but deductive simplification as well. As an example, let us give a careful proof of the chain rule for the derivative of a composite function. This is a theorem whose proof in the classical ε-δ context is unnatural because of the possibility of a certain difference being zero.

THEOREM 6. *Let g be a real function differentiable at x while the real function f is differentiable at $g(x)$. Then the composite fg is differentiable at x and $(fg)'(x) = f'(g(x)) \cdot g'(x)$.*

Proof. Let $i \in I - \{0\}$ be given. For some infinitesimal k,

$$\frac{g(x+i)-g(x)}{i} = g'(x) + k.$$

Thus, the difference

$$j = g(x+i) - g(x) = (g'(x)+k)i$$

is infinitesimal. Similarly,

$$f(g(x)+j) - f(g(x)) = (f'(g(x)) + h)j$$

for some $h \in I$. Thus,

$$\frac{fg(x+i)-fg(x)}{i} = \frac{f(g(x)+j)-f(g(x))}{i} = \frac{(f'(g(x))+h)(g'(x)+k)i}{i}$$
$$= f'(g(x))g'(x) + t$$

where $t \in I$; $f'(g(x))g'(x)$ is thus the real part of the ratio $[fg(x+i)-fg(x)]/i$.

This is far from the only example of deductive simplification resulting from the systematic use of R^*.

†Let us remind ourselves that the points of the real line have a rich structure according to any of the well-known methods of constructing R from Q, e.g., Dedekind cuts or equivalence classes of Cauchy sequences.

6. Conclusions. The "modernization" of the teaching of calculus, implemented in the 1950's and 60's, involved a substantial increase in rigor as well as in linguistic and conceptual sophistication. It is not yet clear whether this really resulted in increased conceptual sophistication on the part of the student, and it almost certainly did not result in increased manipulative ability. I feel that using the hyperreal numbers as a framework for doing calculus holds forth the promise of producing both. The few basic concepts of ring theory necessary to a clear understanding of R^* represent real knowledge and not just an elaborate reformulation in set-theoretical terms of what is already known, as was unfortunately so often the case with some of the "modern" textbooks.

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ANY NEW HELLY NUMBERS?

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We often ask our friends "Any new Helly numbers?" Naturally, we mean "Have you found any new theorems of Helly type relating to families of convex sets?" Since Helly's theorem [5] states that "a family of bounded closed convex sets in Euclidean space E^n has nonempty intersection if and only if every $n + 1$ or fewer members of the family have nonempty intersection," then $n + 1$ is, by definition, a Helly number. It has been our gratifying experience that a casual inspection of a combinatorial situation often reveals some Helly numbers. Also, occasionally a trivial consequence of Helly's theorem immediately suggests a problem which one is unable to solve.

1. For instance, consider a convex polyhedron P in E^n.

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