Philosophy of Logic

Deduction-based approaches

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Through the combinatorial grammar of deduction

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On the game of giving and asking for reasons

- from consecutions to sequents
- from the abstract (Tarski) to Hilbert-Frege systems
- from Hilbert-Frege systems to Gentzen systems
  - from axiom-style to sequent-style, and back
  - from sequent calculus to natural deduction
- other deduction-based approaches

Through the combinatorial grammar of deduction

On the game of giving and asking for reasons

- from consecutions to sequents
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Proof Theory (not 'Syntax'!)

- on the inductively defined set of derivations
- derivations induce consequence relations

Theorem. All logics are presentable as Hilbert-Frege systems.

Some desirable properties

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What and what for?

- purity, simplicity, and whatnot ('harmony'?)
- strategical reasoning, termination
- recursive presentations
- analyticity
- consistency

Some characteristic CPL-derivable sequents

$$\frac{}{\varphi \wedge \psi \rhd \varphi}\ (\texttt{AND1}) \qquad \frac{}{\varphi \wedge \psi \rhd \psi}\ (\texttt{AND2}) \qquad \frac{}{\varphi, \psi \rhd \varphi \wedge \psi}\ (\texttt{AND3})$$

$$\frac{}{\varphi \rhd \varphi \vee \psi}\ (\texttt{OR1}) \qquad \frac{}{\psi \rhd \varphi \vee \psi}\ (\texttt{OR2}) \qquad \frac{}{\varphi \vee \psi \rhd \varphi, \psi}\ (\texttt{OR3})$$

$$\frac{}{\varphi \to \psi, \varphi \rhd \psi}\ (\texttt{IMP1}) \qquad \frac{}{\psi \rhd \varphi \to \psi}\ (\texttt{IMP2}) \qquad \frac{}{\rhd \varphi, \varphi \to \psi}\ (\texttt{IMP3})$$

$$\frac{}{\rhd \varphi, \neg \varphi}\ (\texttt{NEG1}) \qquad \frac{}{\varphi, \neg \varphi \rhd}\ (\texttt{NEG2})$$

$$\frac{}{\rhd \top}\ (\texttt{TOP0}) \qquad \frac{}{\bot \rhd}\ (\texttt{BOT0})$$

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$$\frac{}{\rhd \top}\ (\texttt{TOP0}) \qquad \frac{}{\bot \rhd}\ (\texttt{BOT0})$$

Exercise: Prove the above using the abstract characterizations of the respective connectives.

An H-system for CPL

$$\frac{p_1 \wedge p_2}{p_1}\ (\wedge_1) \qquad \frac{p_1 \wedge p_2}{p_2}\ (\wedge_2) \qquad \frac{p_1, p_2}{p_1 \wedge p_2}\ (\wedge_3)$$

$$\frac{p_1}{p_1 \vee p_2}\ (\vee_1) \qquad \frac{p_2}{p_1 \vee p_2}\ (\vee_2) \qquad \frac{p_1 \vee p_2}{p_1, p_2}\ (\vee_3)$$

$$\frac{p_1 \to p_2, p_1}{p_2}\ (\to_1) \qquad \frac{p_2}{p_1 \to p_2}\ (\to_2) \qquad \frac{}{p_1, p_1 \to p_2}\ (\to_3)$$

$$\frac{}{p_1, \neg p_1}\ (\neg_1) \qquad \frac{p_1, \neg p_1}{}\ (\neg_2)$$

$$\frac{}{\top}\ (\top_0) \qquad \frac{\bot}{}\ (\bot_0)$$

A (simplified) G-system for CPL

Structural rules

$$\frac{}{\varphi \vdash \varphi}\ (init) \qquad \frac{\Gamma_1, \varphi \vdash \Delta_1 \qquad \Gamma_2 \vdash \varphi, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (cut)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ WL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta}\ WR$$

Logical rules

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ (\wedge L_1) \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ (\vee R_1)$$

$$\frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ (\wedge L_2) \qquad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ (\vee R_2)$$

$$\frac{\Gamma_1, \varphi \vdash \Delta_1 \quad \Gamma_2, \psi \vdash \Delta_2}{\Gamma_1, \Gamma_2, \varphi \vee \psi \vdash \Delta_1, \Delta_2}\ (\vee L) \qquad \frac{\Gamma_1 \vdash \varphi, \Delta_1 \quad \Gamma_2 \vdash \psi, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \varphi \wedge \psi, \Delta_1, \Delta_2}\ (\wedge R)$$

$$\frac{\Gamma_1 \vdash \varphi, \Delta_1 \quad \Gamma_2, \psi \vdash \Delta_2}{\Gamma_1, \Gamma_2, \varphi \rightarrow \psi \vdash \Delta_1, \Delta_2}\ (\rightarrow L) \qquad \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \rightarrow \psi, \Delta}\ (\rightarrow R)$$

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ (\neg R)$$

$$\frac{}{\bot \vdash}\ (\bot L) \qquad \frac{}{\vdash \top}\ (\top R)$$

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Logical rules

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ (\wedge L_1) \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ (\vee R_1)$$

$$\frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ (\wedge L_2) \qquad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ (\vee R_2)$$

$$\frac{\Gamma_1, \varphi \vdash \Delta_1 \quad \Gamma_2, \psi \vdash \Delta_2}{\Gamma_1, \Gamma_2, \varphi \vee \psi \vdash \Delta_1, \Delta_2}\ (\vee L) \qquad \frac{\Gamma_1 \vdash \varphi, \Delta_1 \quad \Gamma_2 \vdash \psi, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \varphi \wedge \psi, \Delta_1, \Delta_2}\ (\wedge R)$$

$$\frac{\Gamma_1 \vdash \varphi, \Delta_1 \quad \Gamma_2, \psi \vdash \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \vdash \Delta_1, \Delta_2}\ (\to L) \qquad \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ (\neg R)$$

$$\frac{}{\bot \vdash}\ (\bot L) \qquad \frac{}{\vdash \top}\ (\top R)$$

Exercise: Show that the H-system and the logical rules of the G-system are interderivable.

ND for CPL

Intro & **Elim** rules (framework Set-Fmla)

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ \wedge E_1 \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ \wedge E_2 \qquad \frac{\Gamma_1 \vdash \varphi \quad \Gamma_2 \vdash \psi}{\Gamma_1, \Gamma_2 \vdash \varphi \wedge \psi}\ \wedge I$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee I_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee I_2 \qquad \frac{\Gamma_0 \vdash \varphi \vee \psi \quad \Gamma_1, \varphi \vdash \delta \quad \Gamma_2, \psi \vdash \delta}{\Gamma_0, \Gamma_1, \Gamma_2 \vdash \delta}\ \vee E$$

$$\frac{\Gamma_1 \vdash \varphi \to \psi \quad \Gamma_2 \vdash \varphi}{\Gamma_1, \Gamma_2 \vdash \psi} \to E \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \to I$$

$$\frac{\Gamma_1 \vdash \neg\varphi \quad \Gamma_2 \vdash \varphi}{\Gamma_1, \Gamma_2 \vdash \bot}\ \neg E \qquad \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi} \to I$$

$$\frac{}{\vdash \top}\ \top I \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \delta}\ \bot int \qquad \frac{\Gamma, \neg\delta \vdash \bot}{\Gamma \vdash \delta}\ \bot E$$

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$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee I_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee I_2 \qquad \frac{\Gamma_0 \vdash \varphi \vee \psi \quad \Gamma_1, \varphi \vdash \delta \quad \Gamma_2, \psi \vdash \delta}{\Gamma_0, \Gamma_1, \Gamma_2 \vdash \delta}\ \vee E$$

$$\frac{\Gamma_1 \vdash \varphi \to \psi \quad \Gamma_2 \vdash \varphi}{\Gamma_1, \Gamma_2 \vdash \psi}\ \to E \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ \to I$$

$$\frac{\Gamma_1 \vdash \neg\varphi \quad \Gamma_2 \vdash \varphi}{\Gamma_1, \Gamma_2 \vdash \bot}\ \neg E \qquad \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}\ \to I$$

$$\frac{}{\vdash \top}\ \top I \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \delta}\ \bot int \qquad \frac{\Gamma, \neg\delta \vdash \bot}{\Gamma \vdash \delta}\ \bot E$$

Exercise: Derive the rules of ND from the rules of the G-system.

Beyond the classical case

Beyond the classical case

Capturing other classes of connectives

- positive modal connectives
- negative modal connectives
- restoration connectives
- ... and much more!