Philosophy of Logic

Denotation-based approaches

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The intuition from formal semantics

Look for mathematical **denotations** for the relevant expressions.

On a many-valued notion of logical entailment

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Let there be valuations!

Fix an algebra of sentences $\mathcal{S}$.

A logical matrix $\mathbb{M} := \langle \mathcal{V}, \mathsf{A} \rangle$, is such that:

- $\mathcal{V}$ is an algebra similar to $\mathcal{S}$
- V, the set of *truth-values*, is the carrier of $\mathcal{V}$
- the values in $\mathsf{A} \subseteq \mathsf{V}$ are called *designated*, ('ways of Asserting')
  and those in $\mathsf{E} := \mathsf{V} \setminus \mathsf{A}$ are called *undesignated* ('ways of dEnying')
- $\mathrm{Hom}(\mathcal{S}, \mathcal{V})$ collects all *valuations* on $\mathcal{S}$ induced by $\mathbb{M}$,
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‘Preservation-based’ notions on $\mathcal{S}$ induced by $\mathbb{M}$ (Tarski-inspired)

A compatibility relation:

$\Pi \blacktriangleright \Sigma$ iff $\mathsf{A}_v{:}\Pi$ and $\mathsf{E}_v{:}\Sigma$, for some $v \in \mathrm{Hom}(\mathcal{S}, \mathcal{V})$

A consequence relation on $\mathcal{S}$: $\Pi \rhd \Sigma$ iff it is not the case that $\Pi \blacktriangleright \Sigma$

Set $[\mathsf{A}_v{:}\Psi$ iff $v(\Psi) \subseteq \mathsf{A}]$ and $[\mathsf{E}_v{:}\Psi$ iff $v(\Psi) \subseteq \mathsf{E}]$.

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① Every gcr is characterized by some family of logical matrices.

② A cr $\triangleright$ is characterized by a **truth-functional semantics** (namely, one given by a single logical matrix)

iff it satisfies the following relevance property:

[**Cancellation**] if $\bigcup_{k \in K} \Delta_k \cup \Pi \triangleright \varphi$, then $\Pi \triangleright \varphi$
whenever all sets of sentences from the family $\{\Delta_k\}_{k \in K}$
are pairwise disconnected, no Δ_k is $\triangleright$-trivializing,
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Note

- Not all logics are truth-functional.
- Among truth-functional logics, some logics are not finite-valued.

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③ Every gcr is determined by a bivalent semantics.

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Note 1: This involves a generalization of the notion of compositionality.

Note 2: The algorithm allows for the extraction of a uniform classic-like deductive systems for all the logics to which it applies.

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Conversely:

- axiomatizations may be directly extracted from non-deterministic truth-tabular interpretations of the connectives

misc

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Categoricity: "a logics cannot have two distinct bivalent semantics"

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'The familiar Galois connection between Syntax and Semantics'

For a fixed propositional signature: (check this link)

- the more axioms one adds, the less models one has
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