Philosophy of Logic

Theories, translations, combinations

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Theories are all one needs to know about a certain consequence operator:

- in the framework Set-Fmla [Wójcicki 1988]
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Additional advantage of working with gcrs

Theories that are not finitely axiomatizable using consequence relations may still be finitely axiomatized using generalized consequence relations.

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Some applications

- definitional equivalence
- homophonous translations
- recovering (or not!) a logic inside another
- providing semantics to a given logic characterized by other means

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Some special cases, some difficult problems

- fusions and products of modal logics
- avoiding unwanted interactions
- the semantics of fibring [Caleiro-Marcelino 2023]
  (particularly simple if one uses the Set-Set framework!)