Universidade Federal de Santa Catarina
Centro de Ciências Físicas e Matemáticas
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UFSC MTM

Cálculo 1 (MTM3101 e MTM3110)

Lista 4.9 - Integrais trigonométricas

Última atualização: 22 de junho de 2022.

Exercícios Principais

P1. Calcule as integrais abaixo.

(a) $\displaystyle\int_0^{\pi/2} \cos^5 x\, dx.$ **(b)** $\displaystyle\int \cos^3(3x)\, dx.$ **(c)** $\displaystyle\int \operatorname{sen}^2 x\, dx.$

(d) $\displaystyle\int_0^{\pi} \cos^4(2x)\, dx.$ **(e)** $\displaystyle\int \operatorname{sen}^3 x \cos^2 x\, dx.$ **(f)** $\displaystyle\int_0^{\pi/4} \cos^5 x \operatorname{sen} x\, dx.$

(g) $\displaystyle\int \operatorname{sen}^2 x \cos^2 x\, dx.$ **(h)** $\displaystyle\int \operatorname{sen}^2 x \cos^4 x\, dx.$ **(i)** $\displaystyle\int \operatorname{cotg}^5 x\, dx.$

(j) $\displaystyle\int \sec^3 x\, dx.$ **(k)** $\displaystyle\int_0^{\pi/3} \operatorname{tg}^5 x \sec^4 x\, dx.$ **(l)** $\displaystyle\int \operatorname{tg} x \sec^3 x\, dx.$

(m) $\displaystyle\int_{\pi/4}^{\pi/2} \operatorname{cotg}^4 x \operatorname{cossec}^4 x\, dx.$ **(n)** $\displaystyle\int \operatorname{tg}^2 x \sec x\, dx.$ **(o)** $\displaystyle\int \operatorname{sen}(2x)\operatorname{sen}(5x)\, dx.$

(p) $\displaystyle\int \operatorname{sen}(3x)\cos(5x)\, dx.$ **(q)** $\displaystyle\int_0^{\pi/4} \cos(2x)\cos(7x)\, dx.$ **(r)** $\displaystyle\int_{-\pi/4}^{0} \cos^3(2x)\operatorname{sen}^9(2x)\, dx$

P2. Cleverson, Everton, Giuliano, Luciane e Sonia foram encarregados de resolver a integral

$$\int \operatorname{sen}^3 x \cos^3 x\, dx.$$

A resolução e a resposta de cada um deles estão abaixo.

Cleverson.

$$\begin{aligned}\int \operatorname{sen}^3 x \cos^3 x\, dx &= \int (1-\cos^2 x)\cos^3 x \operatorname{sen} x\, dx \overset{u=\cos x}{=} -\int (1-u^2)u^3\, du \\ &= \frac{u^6}{6} - \frac{u^4}{4} + C = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C.\end{aligned}$$

Everton.

$$\begin{aligned}\int \operatorname{sen}^3 x \cos^3 x\, dx &= \int (1-\operatorname{sen}^2 x)\operatorname{sen}^3 x \cos x\, dx \overset{u=\operatorname{sen} x}{=} \int (1-u^2)u^3\, du \\ &= \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\operatorname{sen}^4 x}{4} - \frac{\operatorname{sen}^6 x}{6} + C.\end{aligned}$$

Giuliano.

$$
\begin{aligned}
\int \operatorname{sen}^3 x \cos^3 x\, dx &= \int (\operatorname{sen} x \cos x)^3\, dx = \int \left(\frac{\operatorname{sen}(2x)}{2}\right)^3 dx \\
&= \frac{1}{8}\int (1-\cos^2(2x))\operatorname{sen}(2x)\, dx \overset{u=\cos(2x)}{=} -\frac{1}{16}\int (1-u^2)\, du \\
&= \frac{u^3}{48} - \frac{u}{16} + C = \frac{\cos^3(2x)}{48} - \frac{\cos(2x)}{16} + C.
\end{aligned}
$$

Luciane. Primeiro observemos que $\operatorname{sen}^3 y = \dfrac{3\operatorname{sen} y - \operatorname{sen}(3y)}{4}$. Usando os dois primeiros passos do desenvolvimento do Giuliano,

$$
\begin{aligned}
\int \operatorname{sen}^3 x \cos^3 x\, dx &= \int (\operatorname{sen} x \cos x)^3\, dx = \int \left(\frac{\operatorname{sen}(2x)}{2}\right)^3 dx \\
&= \frac{1}{8}\int \left(\frac{3\operatorname{sen}(2x) - \operatorname{sen}(6x)}{4}\right) dx = \frac{1}{32}\int (3\operatorname{sen}(2x) - \operatorname{sen}(6x))\, dx \\
&= \frac{\cos(6x)}{192} - \frac{3\cos(2x)}{64} + C.
\end{aligned}
$$

Sonia. Além da expressão apontada pela Luciane para $\operatorname{sen}^3 y$, usaremos também que

$$
\cos^3 y = \frac{3\cos y + \cos(3y)}{4}.
$$

Assim,

$$
\begin{aligned}
\int \operatorname{sen}^3 x \cos^3 x\, dx &= \int \left(\frac{3\operatorname{sen} x - \operatorname{sen}(3x)}{4}\right)\left(\frac{3\cos x + \cos(3x)}{4}\right) dx \\
&= \frac{1}{16}\int (9\operatorname{sen} x\cos x + 3\operatorname{sen} x\cos(3x) - 3\cos x\operatorname{sen}(3x) - \operatorname{sen}(3x)\cos(3x))\, dx \\
&= \frac{1}{16}\int (9\operatorname{sen} x\cos x + 3(\operatorname{sen} x\cos(3x) - \cos x\operatorname{sen}(3x)) - \operatorname{sen}(3x)\cos(3x))\, dx \\
&= \frac{1}{16}\int (9\operatorname{sen} x\cos x + 3\operatorname{sen}(-2x) - \operatorname{sen}(3x)\cos(3x))\, dx \\
&= \frac{1}{16}\int (9\operatorname{sen} x\cos x - 3\operatorname{sen}(2x) - \operatorname{sen}(3x)\cos(3x))\, dx \\
&= \frac{1}{16}\int (9\operatorname{sen} x\cos x - 6\operatorname{sen} x\cos x - \operatorname{sen}(3x)\cos(3x))\, dx \\
&= \frac{1}{16}\int (3\operatorname{sen} x\cos x - \operatorname{sen}(3x)\cos(3x))\, dx \\
&= \frac{3}{16}\int \operatorname{sen} x\cos x\, dx - \frac{1}{16}\int \operatorname{sen}(3x)\cos(3x)\, dx \\
&\overset{(\star)}{=} \frac{3}{16}\int u\, du - \frac{1}{48}\int v\, dv \\
&= \frac{3u^2}{32} - \frac{v^2}{96} + C = \frac{3\operatorname{sen}^2 x}{32} - \frac{\operatorname{sen}^2(3x)}{96} + C,
\end{aligned}
$$

em que em $(\star)$ foram usadas, respectivamente, as mudanças $u = \operatorname{sen} x$ e $v = \operatorname{sen}(3x)$.

Quem está certo? Todos obtiveram a mesma resposta?

Exercícios Complementares

C1. Qual das substituições abaixo transforma a integral $\int \operatorname{tg}^5(2x)\sec^{15}(2x)\,dx$ em uma integral da forma $\int (\alpha u^{14} + \beta u^{16} + \gamma u^{18})\,du$, em que $\alpha, \beta, \gamma \in \mathbb{R}$?

(a) $u = \operatorname{tg}(2x)$.

(b) $u = \operatorname{tg}(x)$.

(c) $x = \operatorname{tg}(2u)$.

(d) $x = \operatorname{tg}(u)$.

(e) $u = \sec(2x)$.

(f) $u = \sec(x)$.

(g) $x = \sec(2u)$.

(h) $x = \sec(u)$.

Além disso, determine α, β e γ.

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Gabarito da Lista 4.9

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Exercícios Principais

P1.

(a) $\displaystyle\int_0^{\pi/2} \cos^5 x\,dx = \frac{8}{15}.$

(b) $\displaystyle\int \cos^3(3x)\,dx = \frac{\operatorname{sen}(3x)}{3} - \frac{\operatorname{sen}^3(3x)}{9} + C.$

(c) $\displaystyle\int \operatorname{sen}^2 x\,dx = \frac{x}{2} - \frac{\operatorname{sen}(2x)}{4} + C.$

(d) $\displaystyle\int_0^{\pi} \cos^4(2x)\,dx = \frac{3\pi}{8}.$

(e) $\displaystyle\int \operatorname{sen}^3 x\cos^2 x\,dx = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$

(f) $\displaystyle\int_0^{\pi/4} \cos^5 x\operatorname{sen} x\,dx = \frac{7}{48}.$

(g) $\displaystyle\int \operatorname{sen}^2 x\cos^2 x\,dx = \frac{x}{8} - \frac{\operatorname{sen}(4x)}{32} + C.$

(h) $\displaystyle\int \operatorname{sen}^2 x\cos^4 x\,dx = \frac{x}{16} + \frac{\operatorname{sen}(2x)}{64} - \frac{\operatorname{sen}(4x)}{64} - \frac{\operatorname{sen}(6x)}{192} + C.$

(i) $\displaystyle\int \operatorname{cotg}^5 x\,dx = \ln|\operatorname{sen} x| + \frac{\operatorname{cotg}^2 x}{2} - \frac{\operatorname{cotg}^4 x}{4} + C.$

(j) $\displaystyle\int \sec^3 x\,dx = \frac{\operatorname{tg} x\sec x + \ln|\operatorname{tg} x + \sec x|}{2} + C.$

(k) $\displaystyle\int_0^{\pi/3} \operatorname{tg}^5 x\sec^4 x\,dx = \frac{117}{8}.$

(l) $\displaystyle\int \operatorname{tg} x\sec^3 x\,dx = \frac{\sec^3 x}{3} + C.$

(m) $\displaystyle\int_{\pi/4}^{\pi/2} \operatorname{cotg}^4 x\operatorname{cossec}^4 x\,dx = \frac{12}{35}.$

(n) $\displaystyle\int \operatorname{tg}^2 x\sec x\,dx = \frac{\operatorname{tg} x\sec x - \ln|\operatorname{tg} x + \sec x|}{2} + C.$

(o) $\displaystyle\int \operatorname{sen}(2x)\operatorname{sen}(5x)\,dx = \frac{\operatorname{sen}(3x)}{6} - \frac{\operatorname{sen}(7x)}{14} + C.$

(p) $\displaystyle\int \operatorname{sen}(3x)\cos(5x)\,dx = \frac{\cos(2x)}{4} - \frac{\cos(8x)}{16} + C.$

(q) $\displaystyle\int_0^{\pi/4} \cos(2x)\cos(7x)\,dx = -\frac{\sqrt{2}}{45}.$

(r) $\displaystyle\int_{-\pi/4}^{0} \cos^3(2x)\operatorname{sen}^9(2x)\,dx = -\frac{1}{120}.$

P2. Todas as resoluções e respostas estão corretas. Como as expressões trigonométricas possuem diversas formas de escrita, então respostas aparentemente diferentes podem estar corretas. A resposta à segunda pergunta, se todos obtiveram a mesma resposta, depende do ponto de vista. Por exemplo, a expressão $\dfrac{\cos^6 x}{6} - \dfrac{\cos^4 x}{4}$ obtida pelo Cleverson é diferente da expressão $\dfrac{\operatorname{sen}^4 x}{4} - \dfrac{\operatorname{sen}^6 x}{6}$ obtida pelo Everton. De fato,

$$\left(\frac{\cos^6 x}{6} - \frac{\cos^4 x}{4}\right) - \left(\frac{\operatorname{sen}^4 x}{4} - \frac{\operatorname{sen}^6 x}{6}\right) = -\frac{1}{12}.$$

Sem a constante C, Cleverson e Everton obtiveram respostas diferentes (mas ambas são primitivas para $\operatorname{sen}^3 x \cos^3 x$). Porém, entendendo que a constante C representa todas as possíveis escolhas de constantes, então podemos dizer que

$$\frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C_1 = \frac{\operatorname{sen}^4 x}{4} - \frac{\operatorname{sen}^6 x}{6} + C_2.$$

Fica de alerta que nesse tipo de exercício a resposta do gabarito pode parecer diferente da sua e mesmo assim ambas estarem corretas.

Exercícios Complementares

C1. Item (e) $u = \sec(2x)$, $\alpha = \frac{1}{2}$, $\beta = -1$ e $\gamma = \frac{1}{2}$.