SECOND EDITION

RtI in MATH

EVIDENCE-BASED INTERVENTIONS

Linda Forbringer • Wendy Weber

An Eye On Education Book

ROUTLEDGE

RtI in Math

Learn how to help K–8 students who struggle in math. Now in its second edition, this book provides a variety of clear, practical strategies that can be implemented right away to boost student achievement. Discover how to design lessons that work with struggling learners, implement math intervention recommendations from the Institute of Education Sciences Practice Guides, the National Center on Intensive Intervention, and CEC, use praise and self-motivation more effectively, develop number sense and computational fluency, teach whole numbers and fractions, increase students' problem-solving abilities, and more! This edition features an all-new overview of effective instructional practices to support academic engagement and success, ideas for intensifying instruction within tiered interventions, and a detailed set of recommendations aligned to both CCSSM and CEC/CEEDAR's High-Leverage Practices to help support students struggling to meet grade-level expectations. Extensive, current examples are provided for each strategy, as well as lesson plans, games, and resources.

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RtI in Math

Evidence-Based Interventions

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Linda Forbringer and Wendy Weber

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Introduction

Students in the United States are struggling in mathematics. Results of the 2019 National Assessment of Educational Progress (NAEP), often called the "nation's report card," show that only 21 percent of 12th grade students are proficient in mathematics, while 40 percent of students scored below the basic level (NCES, 2019.). Although the Every Student Succeeds Act of 2015 states that all children will succeed, clearly many children have not been successful in mathematics.

One model for providing early intervention and support for struggling learners is Response to Intervention (RtI). RtI (also called Multi-Tiered System of Supports or MTSS) is a multilevel prevention system that integrates data-based decision-making, high-quality instruction, and intervention matched to student needs in order to maximize student achievement. The initial focus of RtI was primarily on improving reading achievement, but schools have now expanded RtI to mathematics. While a multitude of books and articles have been written about RtI, most of them describe the RtI process, recommendations for universal screening and progress monitoring, and instruction and interventions for reading. As schools begin to look beyond support in reading, there is a need for resources that address evidence-based interventions for mathematics.

This book is for teacher educators, classroom teachers, special educators, math specialists, math coaches, teacher aides, administrators, related service providers, and other professionals who directly or indirectly support students struggling to master mathematics. We begin with an overview of the RtI process and discuss how to use assessment to make instructional decisions in mathematics. Chapter 3 provides an overview of the evidence-based practices for teaching mathematics in the general classroom and the interventions that support students who are struggling with core concepts. Because a large body of research suggests that careful attention to both lesson design and motivational strategies can significantly improve struggling learners' mathematical achievement, we address these foundational topics in Chapters 4 and 5. In the remaining chapters, we provide a detailed description of interventions to help struggling learners master concepts and operations involving whole numbers and rational numbers. Our goal is to clarify how instruction during interventions differs from the core instruction provided in the general education classroom. We include information about locating effective materials to use during interventions, as well as ideas for adapting and supplementing other available materials to provide the intensive instruction that is necessary to support students who require mathematical interventions. The evidence-based interventions discussed in this book follow the recommendations from the Institute of Education Sciences Practice Guide, *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools,* the National Center on Intensive Intervention, the Council for Exceptional Children, and other national experts. Using these strategies can help prevent difficulties, support struggling learners, and allow *all* students to be successful in mathematics.

1

Overview of Response to Intervention in Mathematics

What Is Response to Intervention?

Response to Intervention (RtI) is an innovative framework for school improvement that is designed to help *all* learners achieve academic and behavioral proficiency. Its goal is to prevent learning difficulties through the use of effective, high-quality instruction, early identification of problems, and tiered intervention services. All students' progress is monitored two or three times each year in order to identify individuals who may need additional support *before* they fail. Students who are experiencing difficulty receive targeted, research-based interventions through a tiered support system. Their progress is monitored frequently, and data from these assessments inform instructional decisions. In addition, districts monitor instructional delivery, assessment, and intervention services to ensure that they are implemented as intended.

In the past 5 years, the terminology to describe RtI has changed nationwide. In a 2019 review of terminology used on the websites of U.S. State Departments of Education, a small number of states (i.e., Texas, New Mexico, and Maine) continue to primarily use the term Response to Intervention. The vast majority of states have moved to using the term Multi-Tier Systems of Support or MTSS to describe a more comprehensive system that encompasses academic, behavioral, and a broader framework for school improvement. Many states use the terms RtI and MTSS interchangeably or have completely switched to using MTSS only. Other education experts note slight differences in these two frameworks. Historically, RtI focused on addressing students' academic deficits in reading and math. As the RtI framework evolved into MTSS, it included behavioral needs and became a more comprehensive school-wide framework for comprehensive and continuous school improvement. In an effort to be inclusive and to honor both the original term RtI and the states that still use that term, this book will include both RtI and MTSS together (RtI/MTSS) when referring to the general framework, and solely RtI when referring specifically to addressing students' needs in mathematics.

The landmark legislation of No Child Left Behind (NCLB, 2001) focused on increased accountability of schools, the use of research-based curriculum, highly qualified teachers, and communication with parents regarding their child's academic proficiency. The reauthorized Individuals with Disabilities Education Improvement Act (IDEA, 2004) also required states to include frequent monitoring of students' progress in the general curriculum as part of, or prior to, the special-education referral process. Both laws emphasize the key principles of RtI/MTSS: progress monitoring; high-quality, research-based instruction; application of a research-based process of problem-solving, and increased accountability. While RtI/MTSS includes a process used to determine students' educational needs, it is not synonymous with the special-education referral process.

In the past ten years, practitioners across the country have worked to refine the general principles and guidelines for implementing RtI/MTSS. In 2015, President Obama signed into law Every Student Succeeds Act (ESSA). This reauthorization of the Elementary and Secondary Education Act (ESEA) focused on closing the achievement gap between students and providing equal access to high-quality instruction through improved assessment and accountability. While the details of implementing a multi-tiered model may vary based on local context, the following critical components are constant: (1) providing evidence-based instruction to all students, (2) using data to guide instructional decision-making and evaluate instructional effectiveness, and (3) using multiple levels of support to provide increasingly intense and targeted interventions for students at risk of academic failure.

Evidence-Based Instruction

A core principle of RtI/MTSS is that all students should receive high-quality instruction using methods that have been validated through rigorous research. In the past, pedagogy was often based on educational theory or educator preferences rather than scientific research. However, researchers have increasingly focused on identifying effective instructional practices. As a result, educators have access to a growing list of instructional procedures and programs that have been shown to significantly increase student learning during rigorous scientific experiments.

Unfortunately, evidence-based pedagogy is sometimes slow to make its way into classrooms. Studies document that many evidence-based strategies are not routinely included in textbooks and teacher guides commonly used in our schools (Bryant et al., 2008; Hodges, Carly, & Collins, 2008; NMAP, 2008). While most publishers provide research and testimonials claiming that their products will achieve miraculous results, many of the studies quoted do not meet the methodological criteria required of high-quality research. To qualify as "evidence-based" research under Every Student Succeeds Act (ESSA), a study must meet rigorous standards, including the use of systematic observation or experiment, measurements or observation methods that have been shown to provide valid and reliable data, and rigorous data analysis. The participants, setting, and methodology must be described in sufficient detail to allow other researchers to replicate the study and compare results. The study must have a large sample size (350 participants), a significantly positive effect, and must be implemented in at least two educational sites. In addition, for an evidence-based practice to be labeled as "evidence-based," the practice should be supported not just by a single study, but by multiple studies that meet the standards of methodological rigor outlined by What Works Clearinghouse, a reliable source of scientific evidence on educational programs, interventions, instructional practices. (What Works Clearinghouse, 2020).

While high-quality research evaluating complete math programs is improving, a significant body of research describes instructional procedures that have been found to be effective for teaching mathematics. These methods have produced significant positive effects in multiple high-quality research studies. The What Works Clearinghouse provides information on programs that have been evaluated with rigorous scientific standards. In Chapter 3, we provide an overview of key evidence-based instructional methods for mathematics. Each strategy is described in greater detail in subsequent chapters. These practices meet the high research standards outlined by the IES/WWC.

Data-Driven Instruction

Data-Driven Instruction (DDI) is a cornerstone of RtI. Assessment data are collected and used to evaluate instructional materials and programs, and to guide instructional decisions for individual students and groups of students. Within an RtI/MTSS framework, three types of assessment occur: (1) universal screening, (2) progress monitoring, and (3) diagnostic assessment.

- **Universal Screening:** universal screening is the first step in the data-collection process. The information obtained allows a district to evaluate the effectiveness of its core instructional program and identify students who are struggling or at risk for mathematical difficulty. Universal screening is usually administered two to four times per year. School personnel may review students' performance on recent state or district tests or may administer a math screening test. The results are used to identify students who are not making adequate progress in mathematics and who need additional support in order to attain mathematical proficiency. The National Center on Intensive Intervention (https://intensiveintervention.org/) provides guidelines for selecting assessments and includes reviews of numerous assessment instruments. The Center on Multi-Tiered System of Supports (www.mtss4success.org) also provides guidance for selecting assessment instruments.
- **Progress Monitoring:** students who are not making adequate progress, as indicated by the universal screening results, receive more frequent progress monitoring. Typically, this involves administering short assessments that can detect small changes in student learning. Student responses to intervention are used to evaluate the effectiveness of the current interventions and guide the decision to either increase or decrease the level of support provided or to maintain or change intervention strategies.
- **Diagnostic Assessment:** diagnostic assessments provide more detailed information about students' strengths and weaknesses in specific skill areas. This formative assessment helps teachers identify the specific mathematical content to be addressed and select appropriate instructional strategies and activities.

Each of these types of assessment will be discussed in greater detail in Chapter 2.

Tiered Support

In an RtI/MTSS model, tiers are used to provide increasingly intensive support for struggling learners. The term "intervention" is used to describe the instructional procedures used to support individuals who have not made adequate progress in the core curriculum. Intervention is "extra help or extra instruction that is targeted specifically to skills that a student has not acquired" (Pierangelo & Giuliani, 2008, p. 80). Generally, RtI/MTSS is described as a three-tier model of support, with each tier representing increasingly intense levels of intervention. The online resources include an overview of the increasing support provided at each tier.

Tier 1: General Classroom

Tier 1 represents the general classroom, where a high-quality, evidence-based core curriculum is delivered according to state standards. Research has not clearly identified an optimal amount of time for Tier I math delivery, but generally 50 to 60 minutes are allocated daily. Teachers are expected to differentiate instruction to meet the needs of students who function at varying levels within the general education classroom. The core curriculum should allow at least 80 percent of students to achieve proficiency in mathematics. Since the goal of RtI/MTSS is to reduce the number of students needing more intensive interventions, at Tier 1, teachers must be proficient in providing high-quality instruction and the evidence-based intervention strategies described in subsequent chapters.

Universal screening is administered at least twice a year, and the results are used to evaluate the effectiveness of the core program and to provide early detection of individuals who may need additional support. Schools typically decide on cut-off scores to determine the level of academic proficiency. Some schools use national norms, but many schools determine their own local norms. Students performing at or above the "proficient" level continue to receive instruction in the general classroom, including differentiated instruction. Students falling below the "proficient" level—that is, those who have not made sufficient progress in the general classroom setting—may require additional instructional support (Tier 2 or Tier 3) depending on the severity of need.

Tier 2: Supplemental Support

Students who have not made sufficient progress receive supplemental support in addition to the math instruction provided in the general classroom. Homogeneous groups of two to five students meet for approximately 30 minutes per day to receive Tier 2 targeted instruction that supplements what they receive in the core curriculum. The content of this supplemental instruction addresses gaps in the students' knowledge or focuses on extended instruction in key concepts. It is not a tutoring session to help students complete the general classroom assignments, but rather an opportunity for students to develop or solidify missing concepts and skills that they will need in order to obtain mathematical proficiency. In the RtI/MTSS model, a multidisciplinary team of professionals uses assessment data to identify students' specific deficiencies and then targets interventions to remediate those deficiencies. While some publishers are promoting a "Tier 2 Program" packaged for all students at a particular grade level who need additional support, such a one-size-fits-all approach is incompatible with RtI's emphasis on data-driven instruction. Seldom will all students who experience difficulty at a given grade level have identical skill deficits. Since the time students spend receiving Tier 2 support might mean time they are out of the classroom and therefore not participating in other instructional activities, it is important to spend this time providing focused support in targeted skills the student needs to learn.

Tier 2 instruction is provided by trained personnel, such as a mathematics coach, general education teacher, or other professional who has received special training. The progress of students receiving Tier 2 services is monitored frequently, and the data are used to determine whether students still require intervention. Once the child makes sufficient progress, Tier 2 support can be gradually faded. About 95 percent of students should make adequate progress through the combination of Tier 1 instruction and Tier 2 support. For students who are still making insufficient progress after receiving Tier 2 services, problem-solving teams design a plan for Tier 3 support that includes more targeted, intensive interventions.

Tier 3: Intensive Interventions

Students who have received high-quality instruction but have not made sufficient progress in Tier 2 need more intensive interventions targeting their individual skill deficits. It is generally expected that, if Tier 1 and Tier 2 are implemented successfully, no more than 5 percent of students will require these intensive interventions. Students receiving Tier 3 services generally meet for a minimum of 50 to 60 minutes per day in addition to the core curriculum. Tier 3 may include one-on-one tutoring, or instruction may be provided for small groups of two or three students who demonstrate similar needs. Data-based Individualization (DBI) is a process used to evaluate the effectiveness of more intensive interventions by using the following steps: (1) intensive instruction/intervention, (2) progress monitoring, (3) diagnostic assessment, (4) intervention changes, and (5) ongoing monitoring, data collection, and evaluation (National Center on Intensive Interventions, 2020). These steps provide frequent monitoring and more in-depth diagnostic assessment to ensure effective intensive intervention/instruction for the students with the greatest need for additional support.

Occasionally students who receive special services may follow a different curriculum in place of the core mathematics program. The decision to remove a child from the core program can only be made by an individualized education plan (IEP) team. Except for students who have an IEP specifying that the child will not participate in core math instruction, all students receive Tier 3 support in addition to the core curriculum.

The method of delivering intensive and individualized support typically includes three tiers of support. Overwhelmingly, most states have adopted a three-tiered model. Data collected at each tier reveal the student's responsiveness to intervention, and these data are used when making decisions about eligibility for special education. However, parents have the right to request a formal evaluation for special education at any point in the process, and the RtI process cannot be used to delay or deny this evaluation.

Models of Implementation

Two approaches to RtI/MTSS are described in the literature: a problem-solving model and the standard treatment protocol. Both approaches provide evidence-based instruction to all students, use data to guide instructional decision-making and evaluate instructional effectiveness, and use tiered support to provide increasingly intense interventions for individuals experiencing difficulty. The two approaches differ in the way instructional interventions are selected for use at each tier.

In the problem-solving approach, a team makes instructional decisions based on the individual student's strengths and weaknesses, as revealed during universal screening and progress monitoring. The team identifies areas in which the individual is struggling and then develops an intervention plan tailored to the student needs. While groups of students with similar profiles are grouped together for instruction, the details of the intervention vary depending on the unique needs and performance data of the groups' members. This approach has been used in schools for more than 20 years and is generally favored by practitioners.

In the standard protocol, school leaders typically decide on a select group of research-based interventions that have been proven to increase student outcomes in specific areas. For example, schools may decide that one particular program that targets basic fact knowledge will be used first for any students needing additional instruction in learning their basic facts. This approach is favored by researchers because using one standardized format helps ensure fidelity of implementation.

Summary

Response to Intervention is an integral part of comprehensive school reform. Schools will continue to improve their ability to meet the learning needs of struggling students by implementing evidence-based instructional strategies and utilizing benchmarking and universal screening as a foundation to data-based decision-making. When schools follow an RtI/MTSS framework, students who struggle in mathematics receive increasingly intensive interventions to supplement the research-based core instruction. In the next chapter, we discuss the use of assessment to make data-based instructional decisions. In Chapter 3, we will define the key components of core mathematics instruction and describe the most effective instructional methods to support struggling students.

2

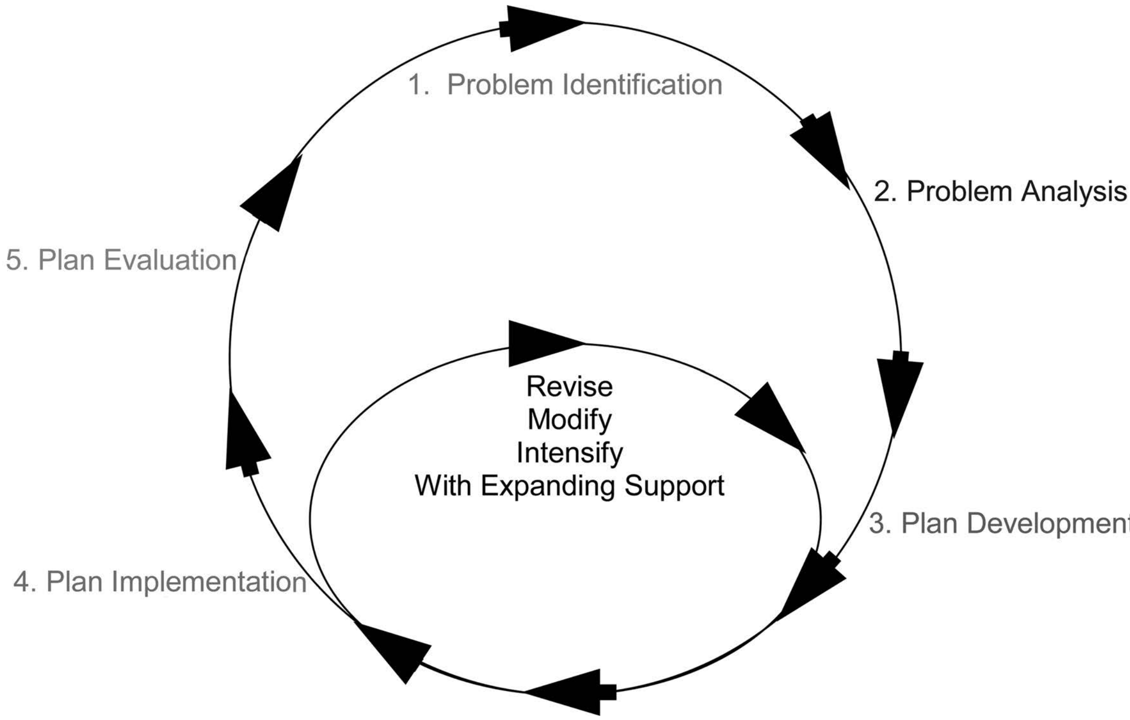
Using Assessment to Make Instructional Decisions

At the heart of Response to Intervention is early identification of students who are at risk of academic failure. The problem-solving model provides an efficient and effective framework to assess students' academic functioning and to use the assessment data to inform and evaluate instructional practices and interventions (Deno & Mirkin, 1977). The five basic steps in the data-based problem-solving model are the following: 1) problem identification, 2) problem analysis, 3) intervention planning, 4) plan implementation, and 5) progress monitoring and plan evaluation. Figure 2.1 gives a visual representation of these important steps developed by Rhode Island Department of Elementary and Secondary Education (2010).

The problem-solving model serves many important functions in a school. First and foremost, it provides an organizational structure that guides teams in their efforts to maximize student success. The five steps mentioned above help teams evaluate school-wide data, prioritize goals, and formulate plans to help all students. The purpose of this chapter is to provide a general overview of the problem-solving process as it relates to instructional decisions in mathematics. There are numerous, high-quality books that go into much greater detail about the technical aspects of educational assessment. See the e-resource for a list of resources containing in-depth information about assessing students' understanding of mathematical skills and concepts.

One of the biggest shifts in current educational practice is the shift to using data to inform instructional decisions in the classroom setting. In the past, providing struggling students with additional support had been heavily dependent on teacher recommendations. Over the past five years, greater emphasis has been placed on using objective academic data to guide instruction and interventions in the classroom. Schools are now using universal screening, benchmarking, and progress monitoring to assess student outcomes, as well as assess the effectiveness of classroom curriculum and instruction. While schools are collecting more data, there is still a gap between collecting data and using data to inform educational decisions. By following the steps of the problem-solving model, educators can ensure that they are identifying and addressing student academic needs in the most targeted and effective way. In this chapter, we will discuss the steps of the problem-solving model as a framework for school-based teams and individual educators to use data to guide their instruction and supplemental interventions.

Figure 2.1 Problem-Solving Model



Source: Rhode Island Department of Elementary and Secondary Education. (2010). *Rhode Island Criteria and Guidance for the Identification of Specific Learning Disabilities*. Providence: Rhode Island Department of Elementary and Secondary Education. Used with permission.

Step 1: Problem Identification

What is the difference between what is expected and what is happening?

To answer this question, we must use the first step in the problem-solving process to identify both those students who are on track and those students who need additional support to be successful. This requires schools to identify local criteria for what is considered adequate performance and the cut-off for what is considered "at risk."

Universal screening provides a comprehensive "sweep" of all children in the school to identify students who need additional support in foundational skills. This sweep enables schools to analyze the effectiveness of the core curriculum and identify which students need additional support.

Screening requires an assessment that is generally inexpensive, easily administered and scored, and provides reliable data on critical skills (number sense, quantity discrimination, etc.). The skills being assessed should have high *predictive validity*, meaning the students' performance on the subskills provides meaningful data regarding future success in that domain. For example, a student who struggles with quantity discrimination and identifying missing numbers is at risk for future challenges in mathematics. Typically, schools conduct school-wide or universal screening two or three times per year. For students who are performing adequately in their classes and on these screening measures, this frequency is sufficient. Other students, who are struggling or who score in the at-risk range on the universal screening, need to be monitored more frequently. This topic will be discussed in more detail later in the chapter under Plan Evaluation.

Core Program Evaluation

One of the main purposes of universal screening is to evaluate the effectiveness of the core curriculum. When schools collect data on all students, rather than analyzing student data in isolation, it is easier to identify trends in student performance across grade levels. Assuming the core curriculum is being implemented with fidelity (meaning all teachers deliver the instruction and curriculum the way they were designed), we can assess how well the curriculum teaches the requisite skills across grade levels and classes. For example, in analyzing the universal screening data for quantity discrimination at the second-grade level at School X, we can see if the curriculum effectively addresses this concept. If we find that a high percentage of students in multiple second-grade classrooms score poorly on the universal screening assessment for quantity discrimination, we could logically deduce that additional time and instruction need to be added to the core curriculum in this specific area. If a small percentage of students score poorly, we can conclude that the core curriculum is adequately covering the concept of quantity discrimination for the majority of the students. It should be noted that the appropriateness or adequacy of the core curriculum also depends on the students receiving instruction in that curriculum. Since students' background knowledge and mastery of skills will vary from year to year, it is possible that the core math curriculum adequately meets the academic needs of the students in some years, but that in other years supplemental instruction or materials may need to be added to the core curriculum. By using universal screening data to assess the effectiveness of the core curriculum, school leaders can ensure that all the students are receiving quality and effective instruction in Tier 1.

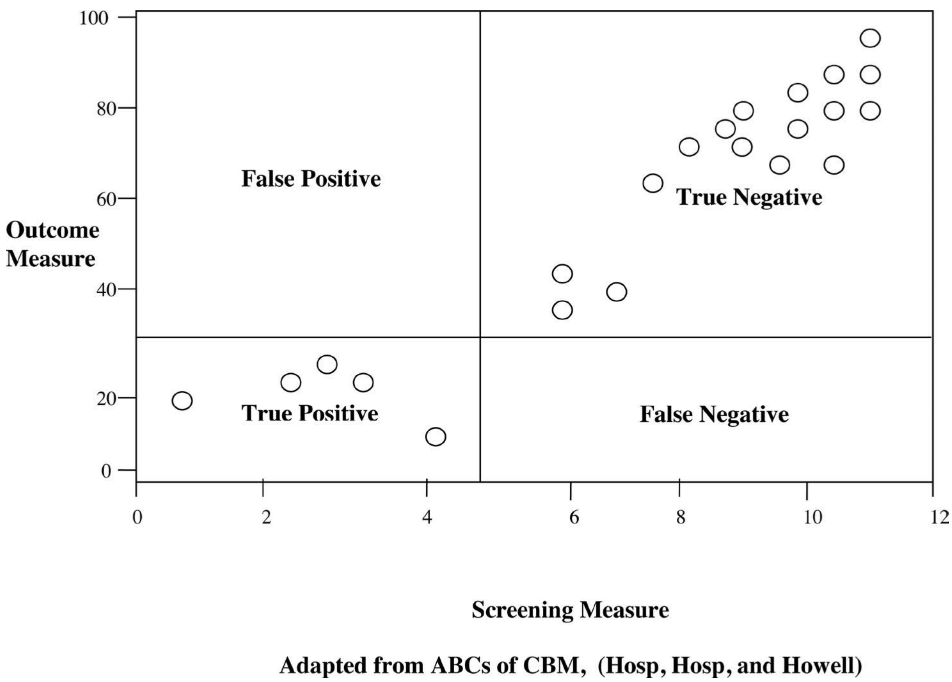
Identifying Struggling Learners

The main purpose of screening all children in the school is to identify the students who are performing adequately and those who are at risk for academic failure. Schools use various assessments to evaluate how students are performing academically. Some examples of universal screeners are curriculum-based measurement (CBM), statewide assessments (Illinois Assessment of Readiness, Texas Assessment of Knowledge and Skills, California Smarter Balanced Summative Assessments), and other informal standards-aligned assessments. While each of these assessments provide valuable information about the student's performance, it is paramount that multiple sources of data are used to determine a student's need for additional academic support. Collecting data from multiple sources allow educators to confirm that an academic issue really exists across settings and time and is not simply a single piece of data that may or may not represent the student's actual academic functioning. Two key features of a universal screening tool are sensitivity and specificity. The *sensitivity* of the screening tool refers to how accurately it identifies students who are at risk (true positives), while the *specificity* refers to how well the tool identifies students who are not at risk (true negatives). In Figure 2.2, an "ideal" screen is depicted by showing that all students who are at risk and all students who are not at risk are accurately identified. Additionally, in the ideal screen, no students are incorrectly identified as being at risk or not at risk. This graph in Figure 2.2 would indicate that the measure is very accurate in identifying struggling learners.

If we can reliably identify students who need additional support and those students who are performing adequately without additional support, we can be more efficient and effective in delivering targeted explicit instruction and interventions.

After educators administer the universal screening and then collect and analyze the student performance data, it is important for teachers to monitor the progress of their students throughout the year. Students who performed adequately on the screening assessment and are considered at or above benchmark only need to be monitored three times per year.

Figure 2.2 The Ideal Screen



Source: Adapted from Hosp, Hosp, & Howell, 2016.

Students who are identified as at risk and require supplemental support in mathematics should be monitored monthly to ensure that the interventions and additional support effectively assist them to make adequate progress toward the benchmark. Those students identified as needing "intensive" support should receive more explicit small-group support in addition to Tier 2 services; these students should be monitored at least every two weeks (ideally weekly). The general rule of thumb about how frequently to monitor student progress is this: the more severe or intensive the need, the more frequently progress should be monitored. The National Center on Intensive Interventions provides information on selecting instruments for universal screening and progress monitoring (https://charts.intensiveintervention.org/chart/progress-monitoring). Its website contains reviews of several assessment measures, which are summarized in an easy-to-read "Tools Chart." Students who are identified as struggling will need additional support. For these students, we move to the next step in the problem-solving process: problem analysis.

Step 2: Problem Analysis

What is the nature of the problem? Why is the problem occurring?

To answer these questions, qualified school personnel must analyze the students' work and possibly do additional diagnostic testing in an attempt to pinpoint the nature of the discrepancy between the students' performance and the expected level. This step in the problem-solving process is crucial because it sets the foundation for the subsequent plan that will identify the students' targeted area of need and also guide the intensity and type of intervention to be matched with that need. Once we have identified the students who performed below the expected benchmark on the school-wide assessment, we use diagnostic assessment techniques to gain in-depth information about the individual student's specific strengths, weaknesses, and instructional needs. Diagnostic assessment can include administering additional assessments, conducting error analysis, observing students while they work, or using structured interviews to gain further insight into their mathematical reasoning. The information gained during diagnostic assessment is used to guide the intervention planning.

Error analysis involves identifying consistent patterns of errors that a student makes. To conduct an error analysis, we collect three to five examples that illustrate the student's work for a particular type of problem, such as adding single-digit numbers or dividing by a one-digit divisor. These samples can be taken from the student's daily work or from responses on the universal screening or other assessment measures. We then analyze the samples, looking for patterns in the errors and identifying possible reasons for the errors, such as lack of understanding of place value or incomplete mastery of basic facts. For example, on a universal screening measure, a student might have missed most of the problems involving addition of one- and two-digit numbers. It would be easy to assume that the student needs an intervention focused on adding one- and two-digit numbers. However, we can use error analysis to confirm this hypothesis or perhaps identify a reason for the errors that would suggest a different intervention. Error analysis helps pinpoint areas of confusion and allows us to select interventions that will most efficiently target the student's needs.

Student errors fall into several categories. The simplest errors involve problems with computational fluency. Basic math facts include the addition, subtraction, multiplication, and division problems formed with two single-digit numbers. Students sometimes miss math problems simply because they made an error at the basic fact level. Look at these two examples from one student's work:

$$\begin{array}{r} \overset{1}{3}9 \\ +\quad 6 \\ \hline 44 \end{array} \qquad\qquad \begin{array}{r} \overset{1}{8}7 \\ +\quad 36 \\ \hline 125 \end{array}$$

In these problems, the student correctly executed the regrouping algorithm but made an error calculating basic facts. In the first problem, she added 9 + 6 and recorded the sum as 14. In the second problem, she added 7 + 6 and recorded the sum as 15. Since she completed the regrouping part of the problem correctly, additional instruction in regrouping might be unnecessary. However, if the student consistently says the sum of 9 + 6 is 14 or the sum of 7 + 6 is 15, she may need additional work to master basic facts. On the other hand, factual errors also occur when students make careless mistakes because their attention has wandered or they are rushing through the assignment. Additional investigation may be necessary to discriminate between these two scenarios. If the problem is due to carelessness, the student can usually fix the error when asked to review the work. A student who struggles when computing basic facts, needs additional practice to develop computational fluency, while a student who makes careless errors will benefit more from an intervention that teaches him to check his work or that rewards computational accuracy. Both students might make errors on the regrouping portion of the screening measure, but neither of these students needs an intervention focused on learning how to regroup in addition.

A more serious type of error occurs when students lack conceptual understanding. Consider how a different student solved the same two problems:

$$\begin{array}{r} 39 \\ +\quad 6 \\ \hline 315 \end{array} \qquad\qquad \begin{array}{r} 87 \\ +\quad 36 \\ \hline 1113 \end{array}$$

These errors do not involve computational fluency. This student correctly added the digits in the ones and tens columns, but recorded the sums without regrouping. Such errors reveal a lack of understanding of place value, which is a far more serious problem than a simple computational error. An intervention for this student would require instructional activities that first develop his understanding of place value followed by instructional activities to develop understanding of regrouping.

Although both of these students made errors on the same problems, error analysis suggests they need very different interventions. It is a common practice to mark math problems correct or incorrect and then to calculate the student's score based on the percentage of correct problems. Students can get an incorrect final answer for a variety of reasons, so it is important for teachers to tease out the type of errors the students make in order to select the appropriate intervention strategies.

While error analysis is a critical component for selecting effective interventions, a study of subtraction error patterns conducted by Riccomini (2005) found that only 59 percent of general education teachers were able to correctly identify error patterns, and even fewer teachers were able to design targeted instruction to address the error patterns they identified. These findings suggest that teachers may need additional training and practice to develop this important skill. An excellent resource for developing skill in error analysis is the book *Error Patterns in Computation: Using Error Patterns to Improve Instruction, 10th edition,* by Robert B. Ashlock.

Students can also make procedural errors by failing to follow the correct steps (or procedures) required to solve the problem. In the following example, the student added from left to right, beginning with the tens column and ending with the ones column.

$$\begin{array}{r} {}^{1} \\ 87 \\ +\quad 36 \\ \hline 114 \end{array}$$

Procedural errors sometimes occur because the student lacks conceptual understanding. In the above example, the student may be confused about place value and so make errors because he is trying to execute an algorithm he does not truly understand. If that is the case, an appropriate intervention would focus on developing understanding of place value and later of the regrouping process. However, the student may have a solid understanding of place value and the regrouping process, but have problems with reversals. A student who occasionally tries to read from right to left may have read the above problem as 78 + 63. Such a student would not need additional instruction in place value or regrouping, but might benefit if an arrow is placed across the top of the page pointing from left to right in order to remind him which way to read the problem. Procedural errors can also occur for a variety of other reasons, such as memory deficits, visual-motor integration problems, and impulsivity. Additional investigation might be needed to determine the cause of the procedural error in order to select an appropriate intervention.

Information gleaned through error analysis can be enhanced through observation and discussion. Interviews provide valuable insight into students' mathematical reasoning and are increasingly advocated to improve mathematical instruction (Allsopp et al., 2008; Buschman, 2001; Crespo & Nicol, 2003; Ginsburg, Jacobs, & Lopez, 1998; Long & Ben-Hur, 1991). One interview technique involves asking the student to "think aloud" while solving the problem. Watching the student work and hearing her thinking may reveal possible misconceptions. Follow-up questions can provide further information. For example, you can ask the student why she selected a particular strategy, ask her to explain her reasoning or

Figure 2.3 Questions to Evaluate Student Understanding

Questions to Evaluate Student Understanding

- Can you explain what you have done so far?
- How did you know to do that?
- Will that work every time?
- Why did you do it that way?
- How did you get your answer?
- Can you explain why that works?
- What should I do next?
- What is the next step?
- What does that that mean?
- How did you reach that conclusion?
- Does your answer make sense?
- Does your answer seem reasonable?
- How would you prove that?
- Can you convince me that your answer makes sense?
- When would we use this?
- What would be another example where we would use this?
- Can you use these blocks to show me what that means?
- Can you make a model to show that?
- Can you draw a picture to show what that means?
- How else could you show that?

suggest alternative approaches, use objects or pictures to demonstrate the solution, or prove that the answer makes sense. Another interview strategy is to let the student play the role of teacher and show you how to solve the problem. Interviews provide evidence of students' mathematical reasoning. They can reveal gaps in learning, provide insight into the thinking strategies students use, and identify the strengths and weaknesses in their understanding. All this information can guide intervention planning. Figure 2.3 shows an example of interview questions that can be used to evaluate student understanding.

Problem analysis helps us develop a complete picture of a student's mathematical understanding. Obtaining this level of detailed information about why the student is struggling allows us to target our instruction or intervention to effectively and efficiently address the specific skills or concepts a student needs to develop. By pinpointing the deficit area, we can identify appropriate instructional goals and interventions that will enable the student to obtain mathematical proficiency.

Step 3: Intervention Plan Development

What is the plan? What is the goal? How will we measure student progress?

After forming a hypothesis about why the student is struggling in math, the intervention team can then devise a plan that matches the intensity and type of intervention that will best meet the student's academic needs. To answer the above questions, the problem-solving team should identify strategies that have research or evidence to support their effectiveness in the target area with a similar population of students. The two most common methods for providing interventions in an RtI framework are to use a standard protocol and to use individualized problem-solving. The biggest difference between the two methods is that educators using standard protocol group students with like needs and implement a standard, research-based intervention that addresses the specific needs of that group of students, while educators using individualized problem-solving consider each student individually. Both methods set goals for the students and use interventions that address the specific needs of those students. Typically, students with severe academic deficits require an individualized plan and may require a variety of specialists to be involved in the decision-making process. It is also common for schools to use standard protocol to address student skill deficits by implementing supplemental interventions in small groups before referring the student for additional support. For example, any students who are struggling with their math facts at a predetermined level might automatically receive additional support by completing computer-assisted basic math fact problems (such as MathFacts in a Flash by Renaissance Learning) in the deficit area. Figure 2.4 outlines the differences between standard protocol and individual problem solving.

Figure 2.4 Comparison of RtI Approaches
A comprehensive school-wide RtI framework includes multiple approaches to providing early intervention for students who are struggling or advanced and not sufficiently challenged. Interventions begin in the classroom at Tier 1. Students not progressing can move to Tier 2 through two options: 1) standard protocol interventions selected by the school to address multiple students' needs, or 2) the problem-solving approach, which is most effective for students with multiple skill deficiencies or complex situations.

Comparison of RTI Approaches		
	Problem Solving	Standard Protocol
Universal Screening	Class-wide assessment/ universal screening is administered to determine the effectiveness of classroom instruction. Struggling readers are identified.	
Tier 1	Frequent progress monitoring is conducted to assess struggling students' performance levels and rates of improvement.	
Tier 2	A team makes instructional decisions based on an individual student's performance. Struggling students are presented with a variety of interventions, based on their individual needs and performance data.	The person delivering the intervention makes instructional decisions that follow a standard protocol. Struggling students are presented with one standard, validated intervention that addresses a variety of skills.
Tier 3	Students whose progress is still insufficient in response to Tier 2 instruction may receive even more intensive instruction. Depending on a state's or district's policies, some students may qualify for special education services based on the progress monitoring data. In some states or districts, students may receive either an abbreviated or comprehensive evaluation for the identification of a learning disability.	

The IRIS Center. (2020). RTI (Part 1): An overview. Retrieved from https://iris.peabody.vanderbilt.edu/module/rti01-overview/

Source: IRIS Module.

Figure 2.5 Math Intervention Plan

Student's Name: John Doe **Referring Teacher: Mrs. Teacher**	
Problem Statement:	**John is able to complete 10 correct digits per minute (in addition, sums to 18). The fall benchmark is 40 correct digits.**
Specific Intervention and General Goal:	**John will practice addition facts on the computer or with a peer helper for 5 minutes daily with flashcards or MathFacts in a flash.**
Length of time:	**4 weeks**
Days per week:	**5 days/week**
Number of Minutes per Day:	**5 minutes/day**
Where intervention will be provided:	**Classroom**
Progress monitoring to determine student progress:	**Timed addition facts on Fridays**

After the student's academic needs are identified, the teacher and/or the problem-solving team determine appropriate goals for the student. Many times, if the area of need is not severe, the team can work backward from the performance level that is considered adequate for future success. After determining an appropriate performance level on a given skill or concept, the team uses local or national norms to identify the level that will enable the student to close the gap between her current performance and her average performing peers. For example, the student may be scoring in the 10th percentile on addition facts. The goal should not be that the student will perform at the 90th percentile, but rather the 50th percentile if that is deemed sufficient performance to progress in the mathematics curriculum and other formal and informal assessments. See Figure 2.5 for an example of an intervention plan for a second-grade student.

Step 4: Plan Implementation

Is the plan being implemented with fidelity?

After the plan has been implemented, it is critical that the fidelity of implementation is documented to ensure that the student is getting the correct intervention content, amount of time, and intensity that was determined to be the best support to address the student's specific academic needs. While there are multiple definitions and contexts for considering the fidelity of implementation, for the purposes of this book, we are referring to the level to which a specific instructional plan is implemented and executed in the way it was designed (i.e., length, duration, intensity, etc. of instructional strategies and interventions). There are many ways to ensure the intervention plan is being implemented with fidelity. Teachers or other support staff can keep a log of daily intervention group skills, time, number of students in the group, and so on. It is also possible to teach students how to monitor their own progress; in some cases, it is appropriate for the student to maintain a log showing the time spent on the specific skill if the process is supervised by a teacher. Some computer programs have developed a way to track when and what students practice when working on computer-assisted programs. Regardless of which option you choose, it is important to be consistent and diligent about keeping accurate records so that when a teacher or team analyzes the student's progress in the targeted skill or concept, there is a record of the prescribed plan being followed. Math curricula and intervention programs are designated "evidence-based" if they have produced significant increases in achievement outcomes in multiple high-quality studies. When a team selects an evidence-based intervention and

Figure 2.6 Questionnaire to Determine the Fidelity of the Intervention Plan

Did the interventionist have adequate training to implement the strategy or intervention the way it was designed?	**Yes**	**No**
Was the intervention provided in the prescribed amount of time each day/week?	**Yes**	**No**
Did the student receive the intervention for the determined length of time (e.g. 6 weeks)?	**Yes**	**No**
Was the quality of delivery consistent and documented?	**Yes**	**No**
Did the interventionist contaminate the prescribed strategy/intervention by piece-milling content from different programs?	**Yes**	**No**
Were students on task and engaged during the intervention time?	**Yes**	**No**
Can it be determined that the student's progress or lack of progress is directly influenced by the prescribed intervention plan?	**Yes**	**No**

implements it in the same way that it was used in the research studies—that is, implements it with fidelity—then their students should obtain similar outcomes. If the intervention is changed, then similar outcomes cannot be assumed. Therefore, if there is empirical evidence that a specific strategy or program produces certain student outcomes, it is imperative that the strategy be used consistently and in the way it was designed. Figure 2.6 provides an example of a questionnaire that could be used to assess whether an intervention was implemented with fidelity. If a student fails to make progress, the questionnaire can help the team determine whether the designated intervention plan was followed consistently. If the plan was followed consistently and the student did not progress, then the team should consider additional diagnostic assessment and/or a new intervention strategy. On the other hand, if the original plan was not followed consistently, then it would be appropriate to continue the intervention but take steps to ensure fidelity of implementation.

Inherent in a plan being implemented with fidelity are the following key aspects: 1) the implementer has adequate training in the method or program, 2) the implementer has access to the needed materials, space, and scheduled time to successfully implement the plan as designed, 3) progress is monitored frequently to assess the student's level of performance and the appropriateness of the intervention in addressing the targeted area, and 4) accurate and consistent documentation of the intervention is maintained throughout the entire plan or until the team decides to make a change.

Step 5: Plan Evaluation

Is the plan working? Is the student making adequate progress? What do we need to maintain or change?

If we use data to identify a student's academic needs and we implement an intervention plan that specifically addresses the areas of need, then we can determine if the plan is successfully targeting the area of need and enabling the student to make adequate progress by closing the gap between the current and the expected level or benchmark. While there are many factors that must be considered when making instructional decisions, an in-depth discussion of those factors is beyond the scope of this book. Figure 2.7 lists additional resources containing information about setting criteria, cut scores, decision points, and so on.

For the purposes of this book, the plan evaluation step in the problem-solving process provides teachers and/or teams with a point at which they can evaluate their own efforts in addressing a student's targeted needs. The plan can be continued as designed (if it is determined to be working but needs to be extended), minimally revised (possibly by increasing the number of times per week), or completely changed (if the student is not responding to the designed intervention plan in its current design).

Figure 2.7 Resources for More Information about Math Assessment

Selecting and Evaluating Assessment Measures:

Center on Instruction.(COI)

www.centeroninstruction.org

COI is a website funded by the U.S. Department of Education that provides a collection of free, scientifically based resources on instruction. One resource available through the site is *Improving instruction through the use of data. Part 1: How to use your data to inform mathematics instruction (2011)*. This professional development module includes a PowerPoint presentation and manual describing progress monitoring in mathematics at the elementary and secondary levels, and as well as a list of additional resources.

The National Center on Multi-Tier System of Supports

https://mtss4success.org/essential-components/progress-monitoring

The National Center on MTSS provides information on selecting instruments for universal screening and progress monitoring. Reviews of several assessment measures are summarized in an easy to read "Tools Chart." This chart is continually updated, so it provides an excellent resource for selecting assessment instruments.

The National Council of Teachers of Mathematics (NCTM)

www.nctm.org

NCTM is the professional organization for teachers of mathematics. Their website offers a wide variety of high-quality publications on the subject of formative assessment in mathematics, as well as other resources for teaching mathematics.

Research Institute on Progress Monitoring

www.progressmonitoring.net

The Office of Special Education Progress funded the Research Institute on Progress Monitoring. The site contains multimedia presentations and technical reports on several assessment measures for mathematics.

Books

Ashlock, R. B. (2009). *Error patterns in computation: Using error patterns to help each student learn*. (10th Edition). Upper Saddle River, NJ: Merrill Prentice Hall.

Figure 2.7 (Continued)

This text provides examples of common errors students make in operations involving whole numbers and fractions, followed by suggestions for effective intervention strategies for each error pattern.

Hosp, M. K., Hosp, J. L., & Howell, K. W. (2016). *The ABCs of CBM: A practical guide to curriculum-based measurement*. (2nd Edition). New York, NY: Guilford Press.

This book provides in-depth information about creating and using CBM for screening and progress monitoring.

Spinelli, C. (2012). *Classroom assessment for students in special and general education, 3rd edition*. Boston, MA: Pearson.

This text contains information on using interviews, observation and questionnaires, and provides sample task analysis checklists.

Ysseldyke, J. E., Salvia, J., & Bolt, S. (2016). *Assessment in Special and Inclusive Education*. (13th Edition). Boston, MA: Houghton Mifflin.

Websites

Math Assessments for Universal Screening & Progress Monitoring

Accelerated Math
www.renlearn.com
This software programs allows educators to monitor individual student progress by creating assessments tailored to the student's current skill level.

AIMSweb
www.aimsweb.com
AIMSweb provides a progress monitoring system in the areas of reading, spelling, written expression and mathematics. It includes a Web-based data management and reporting system.

easyCBM
http://easycbm.com
EasyCBM provides 3 forms of a screening measure for grades K-8 as well as multiple assessment measures for progress monitoring. The Web site provides data management, assessment measures, report generating, and training on administration and scoring.

CBM Warehouse
http://www.jimwrightonline.com/htmdocs/interventions/cbmwarehouse.php

Figure 2.7 (Continued)

This site is devoted to providing CBM resources to do everything from school-wide screening to individual student progress monitoring.

mClass:Math
www.wirelessgeneration.com
mClass:Math is a set of screening and progress monitoring measures for students in Grades K-3.

Monitoring Basic Skills Progress
www.proedinc.com
Monitoring Basic Skills Progress provides CBM assessments for computation and concepts and applications for students in grades 1-6.

National Center on Intensive Intervention
https://intensiveintervention.org/
This website is a comprehensive site for resources and tools for educators working with students who need intensive interventions.

Process Assessment of the Learner (PAL-2)
http://www.pearsonassessments.com/HAIWEB/Cultures/en-us/Productdetail.htm?Pid=015-8661-729
This software can be used to diagnose math problem areas, and can also be used as a progress monitoring tool for grades K-6.

Slosson Diagnostic Math Screener
http://www.slosson.com/onlinecatalogstore_c51693.html
This math assessment can be administered to groups or individuals. Five grade ranges cover math content from grades 1-8.

STAR
www.renlearn.com
STAR Math is a computer-adaptive assessment of general mathematics achievement including computation, mathematic application and mathematics concepts for students in grades 1 to 12.

Websites for Creating CBM Probes and Math Worksheets

AplusMath
http://aplusmath.com/
This website provides free single-skill sheets to help students improve their math skills. It also provides a worksheet generator.

The Math Worksheet Site
http://themathworksheetsite.com

Figure 2.7 (Continued)

This website provides an online math worksheet generator.

SuperKids
http://superkids.com/aweb/tools/math
This website provides a worksheet creator.

Intervention Central
http://interventioncentral.org
Intervention Central provides an early math fluency generator and a math worksheet generator. It also includes ChartDog graph maker.

Summary

This chapter described the problem-solving model as a clear, efficient, and effective framework for applying student data and instructional programming to improve student academic outcomes. The five basic steps of the problem-solving model are the following: 1) problem identification, 2) problem analysis, 3) intervention planning, 4) plan implementation, and 5) progress monitoring and plan evaluation. By using these steps, teachers and other school personnel can identify and target areas of student need. In the next chapter, we provide an overview of evidence-based strategies and programs for use in the core mathematics curriculum and during targeted interventions.

3

Overview of Evidence-Based Practices for Teaching Mathematics

A growing body of research describes practices that are effective for teaching mathematics. Students who are mathematically proficient may succeed without the benefit of high-quality instruction, but best practice is essential for struggling learners. Consistently implementing evidence-based practices is therefore the first step in supporting at-risk students. In this chapter, we will first discuss recommendations for effective mathematics instruction in the core curriculum. Some recommendations for core instruction also apply to students who need additional support, but there are differences as well. In the second half of the chapter, we will discuss recommendations for supporting individuals who struggle with mathematics and require interventions. We will highlight how instruction during interventions should differ from what happens during math instruction in a general education setting. Each of the recommendations for interventions will be discussed in more detail in subsequent chapters. Readers interested in an overview of the recommendations, as well as a discussion of the rationale and research supporting each recommendation, will find this chapter useful. Readers looking for suggestions about how to implement a particular intervention may find it more useful to go directly to the chapter of interest.

Evidence-Based Practices for the Core Curriculum (Tier 1)

The core curriculum is provided at Tier 1; it is the instruction *all* students receive. Anything that happens in the general education classroom is considered to be part of the core curriculum. State standards specify the content to be covered at each grade level. In addition, most districts purchase a program for mathematics that include textbooks and student workbooks, teachers' guides describing instructional activities, and supplemental materials. Such a program is part of the core curriculum. The core also includes anything else that happens in Tier 1. For example, a school may expect teachers to use additional materials or activities to supplement the purchased program, or a teacher may add activities or substitute materials. Core instruction is not one-size-fits-all. In Tier 1, the content, process, and products are differentiated in response to students' interests, learning styles, and academic readiness. Sometimes instruction is provided in a large group setting, and at other times, students work in small groups or independently. Sometimes, students with similar interests or needs are placed together in homogeneous groups, while at other times, heterogeneous groups are used. Flexible grouping allows teachers to match instruction to students' needs. Differentiating instruction provides extra support and practice for students who are struggling, while advanced students can work with more challenging material.

One of the cornerstones of RtI is that all students should receive high-quality instruction. This means that the materials and instructional methods that teachers use should be supported by rigorous research, i.e., have demonstrated effectiveness with the targeted population in controlled experimental studies, and be implemented with fidelity. In 2006, the U.S. Department of Education appointed a National Mathematics Advisory Panel (NMAP) composed of education professionals, researchers with expertise in mathematics and mathematics instruction, and stakeholders, and charged this Panel with reviewing the research on mathematics instruction available at the time, and identifying instructional practices effective for the core curriculum. The Panel's final report, *Foundations for Success* (2008), is available at www.ed.gov/MathPanel. The Panel's recommendations were incorporated in the Common Core State Standards for Mathematics (http://www.corestandars.org/) that were published in 2010. States that do not use the Common Core State Standards also incorporate these recommendations into their own state standards. The recommendations for evidence-based practices in the core curriculum (Tier 1) are summarized below.

Emphasize Critical Concepts

The Panel's first recommendation was that the curriculum for pre-kindergarten through eighth-grade students should be streamlined to emphasize critical topics. International comparisons reveal that high-performing nations typically focus on in-depth coverage of five or six concepts at each grade level. In contrast, U.S. students traditionally covered more than 20 (Schmidt, Wang, & McKnight, 2005). To help students make optimal progress, districts need to select core instructional materials that provide focused, in-depth coverage of the topics emphasized in their state standards, and minimize the time spent on less critical content.

Teach Critical Foundations to Mastery

Critical foundational skills should be taught to mastery by the grades indicated. The Panel specified, "Any approach that revisits topics year after year without bringing them to closure should be avoided" (NMAP 2008 Fact Sheet, p. 1). In the past, many basal math programs in the U.S. have followed a spiral curriculum design, where the same topics are taught for exposure year after year. Teachers report being told not to worry if a student failed to master a particular skill or concept, because it would be re-introduced later. However, some critical skills and concepts are pre-requisites for more advanced mathematics, and students who fail to attain proficiency will struggle with subsequent lessons, falling further and further behind (Porter, 1989). Evidence suggests that the most effective programs avoid introducing topics year after year without closure. When selecting programs for use in their core curriculum, districts are urged to consider a program that teaches foundational skills to mastery.

Balance Conceptual Understanding, Fluency, and Problem-Solving

Historically, mathematics instruction in the United States emphasized computational and procedural fluency. Students learned rote procedures, but often lacked conceptual understanding. As a result, students who might be able to complete a worksheet quickly and accurately were often unable to apply the same skills when they encountered real-life mathematical problem situations. To improve students' mathematical competency, math educators began to emphasize the importance of developing conceptual understanding and teaching mathematics in the context of solving real problems (NCTM, 1989, 2000). This paradigm shift has had many positive effects, but unfortunately some classrooms de-emphasized computational and procedural fluency so much that students failed to become proficient with basic computation. After examining pertinent research, the Panel concluded that conceptual understanding, computational fluency, and problem-solving skills are mutually supportive, and all three are important components of high-quality instruction (NMAP, 2008). These recommendations have been incorporated into the Common Core State Standards (www.corestandards.org) as well as individual state standards. When adopting material for a core curriculum, districts should therefore select programs that balance conceptual understanding, computational and procedural fluency, and problem-solving.

Use a Combination of Teacher-Centered and Student-Centered Approaches

In recent years, mathematics educators have moved towards an inquiry approach that emphasizes conceptual understanding rather than memorizing facts or teaching a particular algorithm. The teacher first engages students in a real-life problem. Rather than telling students how to solve the problem, teachers give students time to explore and work collaboratively to find a solution. The teacher's role is to facilitate student learning and create a classroom environment in which differing mathematical ideas are shared and valued (Hiebert et al., 1997).

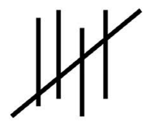
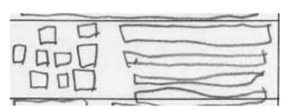
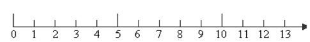
While general education teachers are often taught to use the inquiry method exclusively, special education teachers are taught to use an explicit instruction model of instruction, because extensive research documents the value of explicit instruction for students with disabilities and other struggling learners (McLeskey et al., 2017). Explicit instruction is more teacher-directed. Teachers break skills into small steps and explicitly teach algorithms for solving each type of problem, first modeling a skill and then gradually fading support as students gain proficiency. Since the typical classroom contains a wide range of students, research findings suggest that the core curriculum should not rely exclusively on either instructional model. Instead, districts should seek core curriculum programs that provide a balanced approach to teaching mathematics (NMAP, 2008).

Follow the CPA Sequence

The NMAP (2008) recommended that students at all grades and performance levels experience instruction that incorporates the progressive use of concrete manipulatives, two-dimensional pictorial models, and then abstract symbols. Manipulatives are concrete objects that "appeal to several senses and that can be touched, moved about, rearranged, and otherwise handled by children" (Kennedy, 1986, P. 6). While manipulatives are commonly used by early childhood teachers, they are used less frequently with older students. However, researchers have demonstrated the value of beginning instruction at the concrete level even when working with students in middle school and high school (Butler et al., 2003; Witzel, Mercer & Miller, 2003; Gersten & Clarke, 2010).

Once students demonstrate understanding of concepts using concrete representation, they can progress to using two-dimensional visual representations such as pictures, drawings, number lines, graphs, diagrams, and tally marks to demonstrate mathematical concepts. Abstract presentations using words and symbols to convey mathematical content are most effective after students have developed conceptual understanding using manipulatives and visual representation. This progression is known as the Concrete-Representational-Abstract

Figure 3.1 The Concrete-Pictorial-Abstract Continuum

Concrete	Pictorial	Abstract
• Manipulatives • Act it out	• Pictures • Drawings • Diagrams • Number lines • Tally marks	• Words • Symbols
 	   	$2 + 3$ $a + b$ 5^2 one half

(C-R-A) sequence or the Concrete-Pictorial-Abstract (CPA) sequence (Peterson, Mercer & O'Shea, 1988; Sousa, 2007; Witzel, 2005). Figure 3.1 shows examples of concrete, pictorial, and abstract representation of mathematics. While mathematics educators have long advocated following the CPA continuum (for example, see Van de Walle et al., 2019), many instructional materials rely almost exclusively on abstract representation (Alkhateeb, 2019; Bryant et al., 2008; Hodges, Cady, & Collins, 2008). When a district selects materials that provide sufficient examples of concrete and pictorial representation, it improves all students' understanding.

Select High-Quality Programs and Implement Them with Fidelity

The practices described above have been shown to improve achievement in general education settings. However, in order for students to receive the same positive results obtained in the research studies, each component of the program must be presented as it was designed and presented when the research study results were collected. This means that the program should be used with students who are similar to the students who participated in the study. If a program was effective with normally achieving students, it cannot be assumed that the same program would be equally effective with students who are performing below grade level. In addition, every district has unique demographics, and no single program will meet the needs of all learner populations. For example, students who are transient, come from lower socio-economic backgrounds, have a disability, or who are not native English-language speakers, are at a greater risk for experiencing academic difficulties. A program that is optimal for supporting these students may not be the best choice for high-achieving learners who are not experiencing similar challenges. Learning is optimized when a district selects a core curriculum that matches the learning needs of the majority of its students. When the core program is effectively matched with the student population, about 80 percent of the students should meet benchmarks on state proficiency tests and other screening measures. Less than optimal results can occur when the selected curriculum is not supported by rigorous research, is not a good match for the school's population, or is not implemented with fidelity. If screening data indicate that more than 20 percent of students in a school are struggling, then it is time to re-examine the core curriculum. Solutions include providing additional training and support for teachers, increasing the amount or type of differentiation, or selecting different core curriculum materials.

In order to help schools find high-quality instructional programs, several organizations maintain a database summarizing the research on commercially available instructional materials. The What Works Clearinghouse, which is a division of the Department of Education's Institute for Education Sciences (IES), assesses the rigor of evidence regarding programs, practices, and products, and provides a summary of their findings that can help educators make informed decisions (WWC, https://ies.ed.gov/ncee/wwc/Math/). For basal mathematics programs, each review contains an overview of the program, a summary of research on the program, and a statement about the program's effectiveness. Another website that provides information about the strength of evidence supporting elementary and middle school math programs is The Best Evidence Encyclopedia (www.bestevidence.org). This web site, created by Johns Hopkins University School of Education's Center for Data-Driven Reform in Education and funded by the Institute of Education Sciences, U.S. Department of Education, currently provides ratings for a variety of math programs. These websites continue to update their recommendations to reflect ongoing research findings.

Even when the core curriculum is well matched to the student population and implemented with fidelity, it is expected that up to 20 percent of students may need additional support. In the next section, we describe how instruction should differ for these students.

Interventions to Support Students Who Struggle in Mathematics (Tiers 2 & 3)

The term "intervention" is used to describe instructional activities that provide additional support to improve outcomes for students who are struggling to master the core curriculum. Some recommendations for instruction during interventions are similar to the recommended practices for the core curriculum. However, best practices during interventions include some significant differences which affect both what is taught and how it is taught. Differences between the instruction recommended for the general population and strategies that support learners who struggle to master mathematics are summarized in several publications from IES and leading educators: (Baroody, Burchinal, Carver et al., 2013; Gersten et al., 2009; McLeskey et al., 2017; Powell and Fuchs, 2015; www.centeroninstruction.org/). Recommendations for evidence-based interventions in mathematics found in the Practice Guide are listed in Figure 3.2. This book is based on recommendations from all these sources. We provide an overview of the recommendations below, and explore them in greater detail in subsequent chapters.

Use Explicit, Systematic Instruction

For students who struggle with mathematics, studies show that learning increases when teachers use systematic, explicit instruction (McLeskey et al., 2017). In Tier 1, experts recommend using a balanced approach that combines inquiry methods and explicit instruction, as discussed previously. However, because struggling learners benefit from more extensive

Figure 3.2 Recommendations from the IES Practice Guide: Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools

1. Screen all students to identify those at risk for potential mathematics difficulties and provide interventions to students identified as at risk.
2. Instructional materials for students receiving interventions should focus intensely on in-depth treatment of whole numbers in kindergarten through grade 5 and on rational numbers in grades 4 through 8. These materials should be selected by committee.
3. Instruction during the intervention should be explicit and systematic. This includes providing models of proficient problem solving, verbalization of thought processes, guided practice, corrective feedback, and frequent cumulative review.
4. Interventions should include instruction on solving word problems that is based on common underlying structures.
5. Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.
6. Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.
7. Monitor the progress of students receiving supplemental instruction and other students who are at risk.
8. Include motivational strategies in tier 2 and tier 3 interventions.

Source: Gersten et al., 2009, p. 6.

use of explicit instruction, the Council for Exceptional Children, the National Center on Intensive Intervention, the National Mathematics Advisory Panel, the *High Level Practices for Special Education*, the What Works Clearinghouse Practice Guide, and *High Leverage Practices in Special Education* all recommend using explicit instruction during more intensive interventions (Gersten et al, 2009; Jayanthi, Gersten & Baker, 2008; McLeskey et al, 2017, McLeskey et al, 2017; NMAP, 2008). Students receiving core instruction should experience some explicit instruction at Tier 1 before being moved to Tier 2, but at higher tiers, explicit instruction should be the predominant instructional method. Although systematic, explicit instruction is not a specific mathematics strategy, it may be the most important support interventionists can provide. Chapter 5 contains detailed description of how to use systematic, explicit instruction to support students who struggle with mathematics.

Use Visual Representations

Students who struggle with mathematics have difficulty understanding abstract symbols (Hecht, Vogi, & Torgesen, 2007; van Garderen, Scheuermann, Poch, & Murray, 2018). In order to develop conceptual understanding, they need to first experience mathematical concepts by dramatizing problems or using manipulatives to demonstrate them. After they have mastered a skill using manipulatives, students are ready to use two-dimensional representations such as pictures, tally marks, or graphic representations to help solve problems. Finally, they move to the abstract phase and fade concrete and pictorial representation, instead relying on numbers and symbols to solve problems.

As we discussed earlier, all students benefit when instruction follows the CPA continuum, but it is critical during interventions. Unfortunately, studies show that typical textbooks do not provide adequate concrete and pictorial models of mathematical problems, but instead rely primarily on abstract words and symbols (Alkhateeb, 2019; Bryant et al., 2008; Hodges, Cady, & Collins, 2008). While students who are talented in mathematics may master new concepts, skills, and procedures with minimal concrete and visual representation, the omission of these foundational activities can be devastating for students who require mathematical interventions. In addition, recommendations for pacing instruction for students who struggle with mathematics differ from recommendations for Tier 1 (core) instruction. Multiple studies have demonstrated that students with disabilities or who struggle with mathematics typically need about three lessons at the concrete level, each consisting of approximately 20 problems, before they are ready to fade concrete support, then three 20-problem lessons at the pictorial level before they have developed conceptual understanding and are ready to work solely with abstract words and symbols (for example, see Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Mercer & Miller, 1992; Miller, Harris, Strawser, Jones, & Mercer, 1998). Even when materials designed for core instruction include concrete and pictorial representation, they seldom spend enough time at each level before they drop these visual representations and expect students to work completely at the abstract level.

Students who struggle with mathematics also need to have the relationship between the concrete, pictorial, and abstract depictions explicitly demonstrated. The IES Practice Guide states, "We … recommend that the interventionists explicitly link visual representation with the standard symbolic representations used in mathematics" (Gersten et al., 2009, p 31). In other words, teachers should explicitly teach concepts and operations at the concrete level, then repeat the instruction using the same language and procedural strategies at the pictorial and abstract levels, so that students clearly see the relationship between the various forms of representation.

These recommendations apply at all grade levels because the CPA sequence has been shown to be effective with both elementary and secondary students. In Chapter 6, we discuss the CPA continuum in greater depth. In Chapters 7 and 8, we provide ideas for how to use the CPA continuum to develop conceptual understanding of whole numbers, and in Chapter 10, we apply the sequence to rational numbers.

Focus Intensely on Whole Numbers and Rational Numbers

In the general education classroom, the core curriculum addresses the full range of mathematical content, including number sense and operations, algebra, geometry, measurement, and data analysis and probability. In response to concerns about the scope of our curricula, educators have attempted to limit the range of topics covered each year, encouraging schools to place more focus on in-depth coverage of fundamental concepts and skills. Focusing on in-depth coverage of foundational content is even more important for students who struggle with mathematics. The IES Practice Guide recommends, "Instructional materials for students receiving interventions should focus intensely on in-depth treatment of whole numbers in kindergarten through grade five and on rational numbers in grades four through eight" (Gersten et al., 2009). This means that in the early grades, interventions provided for students receiving Tier 2 and Tier 3 support should emphasize counting, number value, place value, and operations with whole numbers. Topics such as geometry, measurement, and data analysis are important, but most students who receive interventions also participate in Tier 1 instruction, and so will be exposed to that content during core instruction. Because understanding whole numbers and rational numbers forms the foundation for all other mathematics, it is recommended that intervention time focus on these foundational concepts. We will discuss strategies for developing proficiency with whole numbers in Chapters 7, 8, and 9. Once students have mastered this content, the focus should shift to rational numbers, including understanding the meaning of fractions, decimals, ratios and percent, and operations using rational numbers. In Chapter 10, we will discuss interventions for developing proficiency with rational numbers. Because of its intense focus on foundational skills, Tier 2 and 3 instruction will not necessarily align with the content being presented in the general classroom. However, since Tier 2 instruction is provided in addition to Tier 1 instruction, students receiving tiered support will still be exposed to the additional content when they are in the general education classroom.

Build Fluent Retrieval of Basic Facts

All students should master basic math facts involving addition, subtraction, multiplication, and division. Mastery requires students to first develop conceptual understanding of the meaning of the operations. Next, students need to become proficient with computational strategies for solving basic fact problems. Finally, they need to develop automaticity with basic facts. Just as good readers move beyond sounding out words, and instead are able to automatically recognize basic vocabulary, successful math students automatically recognize basic math facts. Once students can compute fluently, they are able to direct their cognitive energy to more complicated tasks. Students who cannot identify basic facts quickly and easily must devote too much of their working memory to computation, which limits their ability to attend to other aspects of instruction. For example, a student might be watching a demonstration on how to regroup when solving a multi-digit subtraction problem, but if the student is unable to solve basic subtraction facts automatically, he or she might have to focus so much energy on the basic computation that the new information about regrouping would be lost.

Research has shown that automaticity with basic facts predicts performance on general mathematics tests (Stickney, Sharp, & Kenyon, 2012), and that students who struggle in mathematics typically lack automaticity with basic facts (Baker & Cuevas, 2018; Gersten et al., 2009). Therefore, the IES Practice Guide recommends, "Interventions at all grade levels should devote about ten minutes in each session to building fluent retrieval of basic arithmetic facts" (Gersten et al., 2009). We describe strategies for developing fact fluency in Chapter 9.

Teach Students to Use Underlying Structures to Solve Word Problems

The purpose of learning mathematics is to be able to apply that knowledge to solve real-life problems. The National Council of Teachers of Mathematics (NCTM), the NMAP (2008), and the Common Core State Standards for Mathematics (CCSSM, 2010) all identify problem-solving as one of the key components of an effective core mathematics program. Unfortunately, students who struggle with mathematics have been shown to experience extreme difficulty solving mathematical word problems (Gersten et al., 2009; Pfannenstiel, Bryant, Bryant & Porterfield, 2015; Jitendra et al., 2015; Stevens & Powell, 2016).

Math programs designed for use as a core curriculum in the general education setting frequently teach problem-solving using a variation of Polya's (1945) four-step process: (1) understand the problem; (2) devise a plan; (3) carry out the plan; (4) look back and reflect. While these are good broad steps, students who struggle with mathematics often become stuck at step one and therefore struggle to devise a plan to solve the problem. Giving them a more detailed step-by-step plan for problem-solving can improve achievement (see, for example, Dennis, 2015; Fuchs, Powell, Cirino, Schumacher, Marrin, Hamlett, & Changas, 2014; Powell & Fuchs, 2018).

A strong body of evidence demonstrates that teaching students to use underlying structures to understand word problems and develop a solution strategy can produce significant improvement in these students' problem-solving performance. When focusing on underlying structures, students first learn to recognize problem patterns, and then learn to organize the information from the problem on the appropriate schematic diagram. Next, students learn a strategy for solving each type of problem. Because the effectiveness of using underlying structures to teach problem-solving is well documented (see, for example, AMTE, 2017; Baroody et al, 2013; Woodward et al., 2012), it is a beneficial approach to use in all tiers. It is critical for students who require additional support. The IES Practice Guide recommends, "Interventions should include instruction on solving word problems that is based on common underlying structures" (Gersten et al., 2009). In Chapter 12, we provide detailed explanation and examples of how to use underlying structures to teach problem-solving.

Include Motivational Strategies

Students who struggle with mathematics often have processing problems, cognitive disabilities, problems with memory retrieval or storage, attention deficits, and other problems that make it difficult to focus on instruction or to complete assignments. In addition, many have experienced failure or frustration when they have attempted mathematical tasks in the past, and so now approach mathematics with trepidation. Therefore, effective interventions must address student motivation. While a discussion of motivation may seem out of place in a mathematics book, research studies demonstrate that the planned use of motivational strategies has a greater impact than the choice of textbook or the use of technology to improve learning outcomes (see for example Epstein et al., 2008; Fuchs et al., 2005; Marzano, Pickering & Pollack, 2001). For this reason, the IES Practice Guide recommends that motivational strategies be included in all Tier 2 and Tier 3 interventions (Gersten et al., 2009). Because motivation is a pre-requisite for all learning, we begin our discussion of interventions by addressing motivation in Chapter 4.

Intensify Instruction

Many publishers offer math programs that they say are designed for use in Tier 2 interventions. Some of these are "validated programs," which means "there is positive evidence, collected during at least one well-conducted randomized control trial, that the program improves the mathematics outcomes of students with MD (mathematical disabilities) in a Tier 2 intervention" (Powell & Fuchs, 2015, p. 183). If a program is validated, then it is appropriate to use during Tier 2 interventions. The National Center on Intensive Intervention provides an Academic Intervention Tools Chart (available online at www.intensiveintervention.org) which summarizes efficacy studies of mathematics intervention programs. This chart can assist educators in finding effective intervention materials.

However, many teachers who provide Tier 2 support do not have access to a validated program that meets the needs of their students (Powell & Fuchs, 2015). When the available materials are not specifically designed for students with math disabilities, then interventionists will need to make adaptations to intensify them. "Intensifying instruction" means adapting the existing program to more effectively address a student's targeted needs. If an interventionist does not have access to a validated program, then the available resources must be intensified in order to incorporate appropriate Tier 2 interventions. The National Center on Intensive Interventions provides a Taxonomy of Interventions that offers assistance on intensifying instruction (https://intensiveintervention.org/taxonomy-intervention-intensity).

Students who need additional support beyond Tier 2 can receive "individualized support" in Tier 3. In this context, the term "individualized" does not mean that instruction must be provided in a one-on-one setting. The term "individualized instruction" or "specially designed instruction" means that instructional objectives and methods are individualized to meet the unique needs of the learner. In other words, the instruction provided at Tier 2 should be further intensified. If a validated program was provided in Tier 2, then Tier 3 interventions may be developed by building on the existing program. If a validated program was not used in Tier 2, teachers must further intensify materials to develop effective individualized interventions at Tier 3 (McInerney, Zumeta, Gandhi, & Gersten, 2014). We will discuss methods for intensifying instruction throughout this book. Additional resources to help interventionists meet their students' needs include the IRIS module, "Intensive Intervention Part 1: Using Data-Based Individualization to Intensify Instruction," (https://iris.peabody.vanderbilt.edu/module/dbi1/) and the National Center on Intensive Intervention website (https://intensiveintervention.org/), which both provide recommendations for intensifying instruction across content areas.

In this chapter, we have described six evidence-based practices for supporting students who struggle with mathematics. If the available materials do not include these practices, then the first step in intensifying instruction is to add one or more of the practices described above. For example, expanding the using of visual representation would be one way to intensify instruction. Abundant research suggests that students who struggle with mathematics benefit from systematic, explicit instruction, so if the available materials do not use systematic, explicit instruction, then making the lesson more explicit would be an effective way to intensify instruction. The instructor might break objectives into smaller pieces, add active review activities before introducing new content, model procedures, provide additional examples, or expand guided practice activities. We will provide a detailed description of the elements of systematic, explicit instruction in Chapter 5. Increasing motivational strategies is another way to intensify instruction. We discuss motivation in Chapter 4. Focusing instruction on foundational content such as whole numbers and rational numbers, basic facts, and problem-solving would be other ways to intensify instruction. We will describe additional ideas for intensifying instruction in subsequent chapters.

Summary

If American children are to become mathematically proficient, they need high-quality instruction. The core curriculum provided to all students in the general education classroom (Tier 1) should use materials and instructional approaches that have been validated through rigorous scientific study and found to be effective with the general education population. Research on effective approaches for teaching mathematics continues to emerge, but a number of evidence-based practices have already been identified that can guide instructional decision-making. The core mathematics curriculum should be streamlined to emphasize critical concepts, establish clear benchmarks that specify the grade by which students should master each concept, and emphasize conceptual understanding, computational and procedural fluency, and problem-solving skills. When selecting mathematics programs to use in their core (Tier 1) curriculum, districts will need to focus on materials that align with their state standards and follow these research-based guidelines, and that have been found effective with students similar to those in the district's general education classrooms. Additional guidelines apply for students who require Tier 2 or Tier 3 support. While struggling learners will be exposed to the full core curriculum in the general education classroom, the interventions they receive through Tier 2 and Tier 3 support should focus intensely on the most critical foundational concepts and skills. These include whole numbers and rational numbers, computational fluency with basic facts, and problem-solving. While all students benefit when concepts are introduced using concrete and visual representation, it is critical for students who struggle with mathematics. In addition, research findings consistently show that students who struggle with mathematics benefit from systematic, explicit instruction and other strategies that intensify instruction. When selecting materials for use in Tier 2 and 3 interventions, districts need to select evidence-based and validated programs that have been found effective with students who require math interventions. Although instructional design and motivation are not topics unique to mathematics, the strong body of evidence supporting these strategies led the What Works Clearinghouse to include them in their recommendations for supporting learners in K–8 mathematics (Gersten et al., 2009), and we therefore focus on these strategies in the next two chapters. In Chapters 6-11, we will discuss how to incorporate other evidence-based recommendations when providing interventions for whole numbers, rational numbers, and problem-solving.

4

Setting the Stage: Increasing Motivation

Why Focus on Motivation?

Motivation is critical for success. Robert Sternberg, former president of the American Psychological Association, describes it as "indispensable" for learning. "Without it," he writes, "a student never even tries to learn" (Grove, 2019; Saeed & Zyngier, 2012; Sternberg, 2005, p. 19). Unfortunately, many of the students who struggle with mathematics lack the motivation to successfully engage in mathematical activities; they dread tasks involving mathematics because they fear another experience of failure (Gersten et al., 2009). Many are struggling to overcome a variety of very real challenges that negatively impact their ability to achieve mathematical proficiency, including language deficits, processing problems, memory and attention deficits, cognitive disabilities, and problems with executive functioning, so their fear of failure may be well-founded. These students may actually need to work harder than their normally achieving peers just to achieve minimal success. Students who already dislike or fear mathematics may find it hard to summon the necessary effort to succeed in mathematical tasks.

It is not surprising, therefore, that research studies find that effective interventions address motivational factors in addition to mathematical content. In fact, studies show that addressing students' motivation, especially with the use of structured rewards, has a greater impact on mathematical achievement than the choice of textbooks or the provision of computer-assisted technology (Best Evidence Encyclopedia, 2020). These findings have led experts to recommend that mathematical interventions should include a motivational component (Gersten et al., 2009; NMAP, 2008).

Motivation can be addressed indirectly through lesson design or more directly using strategies such as self-monitoring, goal-setting, praise, and rewards. We will briefly review some of the components of lesson design that can increase motivation, and then discuss direct motivational strategies in more detail.

Increasing Motivation by Creating Meaningful, Engaging Lessons

Attention is necessary for learning. When students are not attentive and engaged, they are not learning. In order to hold students' attention, math lessons must help students perceive that the information is meaningful to them. The human brain is constantly bombarded with information, and attending to the full deluge of stimuli would be overwhelming. Therefore, the brain is designed to filter incoming information and selectively focus on those stimuli that hold personal meaning. As Wolfe explains in her book, *Brain Matters: Translating Research into Classroom Practice*, "Our species has not survived by attending to and storing meaningless information" (Wolfe, 2010, p. 86). Too often students perceive mathematics lessons as containing meaningless information, and they ask, "Why do we have to learn this? When will we ever use this?" If teachers can relate mathematical content to students' previous experiences and personal interests, students are more likely to perceive the information as meaningful and relevant and so will pay closer attention. The National Council of Teachers of Mathematics (NCTM) places meaning squarely at the center of an effective mathematics program (NCTM, 2000).

Unfortunately, despite the NCTM's emphasis on creating authentic connections, most textbooks only occasionally create a meaningful connection to real-world applications or relate new mathematical concepts to students' personal experience (Hodges, Cady, & Collins, 2008). To increase motivation, educators will often need to add examples of real-world applications for mathematical content. For example, students will be more engaged in finding the diagonal of a rectangle if they realize that it is the dimension used to designate the size of a television screen; they will be more interested in working with fractions if the fractions are related to the beat in their favorite song. When the teacher can successfully combine the students' own interests with relevant mathematical content, their natural motivation is aroused.

Humans are social creatures, so activities that involve social interaction often elicit increased participation and engagement. Even when the topic is not inherently interesting, when the learning process involves social interaction, students may be motivated to attend and participate because they enjoy the peer interaction. This positive emotional response can actually increase learning. Wolfe explains, "The brain is biologically programmed to attend first to information that has strong emotional content … It is also programmed to remember this information longer" (Wolfe, 2010, pp. 88). Using games and other interactive activities can arouse emotional responses that improve long-term learning. In Chapter 8, we will describe several games that can be used to develop computational fluency. On its Illuminations website (illuminations.nctm.org), NCTM provides many additional games that can be used in tiered interventions to increase student motivation.

Cooperative learning activities are another excellent way to incorporate social interaction into tiered interventions. Suggestions for how to begin implementing cooperative learning activities in the classroom are available at Kagan Cooperative Learning (https://www.kaganonline.com/free_articles/research_and_rationale/330/The-Essential-5-A-Starting-Point-for-Kagan-Cooperative-Learning). A variety of formats for peer-tutoring have been developed, all involving students working in pairs or small groups to help each other master the material. The most effective peer-tutoring programs combine highly structured cooperative learning strategies with a structured reward system. We will discuss how to create effective structured reward systems later in this chapter. See the online resource (Figure 4.1) for a description of peer-tutoring programs that have shown strong evidence of effectiveness in mathematics.

Figure 4.1 Go Online Evidence-based Peer-Tutoring Programs for Mathematics

Classwide Peer Tutoring (CWPT)

CWPT combines reciprocal peer tutoring with an incentive system. It is designed to be used in grades K-6 for 30 minutes per day, 4 days a week. Each week students are pretested on that week's content, and assigned to a partner and team for the week. Partners take turns tutoring and testing each other using a highly structured tutoring procedure. Students receive points for answering correctly, for correcting and practicing after an incorrect response, and for weekly test performance. Bonus points are awarded when students are observed engaging in appropriate task-related behavior. Teams earning the greatest point totals receive recognition.

- For additional information, see https://www.rand.org/pubs/tools/TL145.html

Peer Assisted learning Strategies (PALS)

PALS is a version of Classwide Peer Tutoring used to reinforce material previously covered by the teacher. It is designed to be implemented for 25-35 minutes 2-4 times per week. Teachers pair lower and high performing students, and each set of partners works on activities matched to the particular problems they are experiencing. Partners change weekly so that all students have the opportunity to act as both coach and as player. Scripted lessons and materials are available for mathematics content covering kindergarten through sixth grade skills.

- PALS materials can be purchased from https://frg.vkcsites.org/what-is-pals/pals_math_manuals/

Student Teams-Achievement Divisions (now Power Teaching: Mathematics)

STAD is a cooperative learning strategy in which small groups of learners with different levels of ability work together to accomplish a shared goal, and receive recognition for their progress. Students within a class are assigned to 4-5 member heterogeneous learning teams. After the teacher introduces the material, team members help each other master the material with the aid of worksheets, tutoring, discussions and quizzes. At the end of the week each student is quizzed individually. Students are graded on their improvement, and teams with the highest scores and greatest improvement are recognized.

- Power Teaching: Mathematics can be purchased from www.successforall.org

Team Accelerated Instruction: Math (TAI Math)

TAI Math combines interactive instruction by teachers with cooperative learning, individualized instruction, and an incentive system. It is designed to provide supplementary math support for students in grades 3 through 6, or older students who are not yet ready for algebra. Students work in small groups. They complete a pretest, and then are placed in an individualized instruction sequence and allowed to proceed at their own pace. Within their teams, students assist one another with problems and check each others' work. Teachers provide direct instruction to small groups of students who are performing at similar levels. Students receive weekly awards based on the average performance of their teams.

- For additional information, see the following website: https://www.rand.org/pubs/tools/TL145.html
- TAI Math materials can be purchased from https://www.charlesbridge.com/collections/t-a-i-mathematics

Using Self-Monitoring and Goal-Setting to Increase Motivation

Self-monitoring is a strategy that involves students monitoring their own behavior and recording the results. Studies have shown that students who use self-monitoring are more engaged and more productive, have greater accuracy, and show increased awareness of their own behavior (Carr, 2014; Falkenberg & Barbetta, 2013; Schulze, 2016). The IES Practice

Figure 4.2 Self-Monitoring

Self-monitoring teaches students to develop greater metacognitive awareness by learning to assess their own behavior and track the results. Studies of self-monitoring have demonstrated a variety of benefits, including increased self-awareness, fewer disruptive behaviors, increased on-task behavior and increased academic achievement. Self-monitoring and goal setting are evidence-based practices recommended by the What Works Clearinghouse Practice Guide, *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools* (Gersten et al., 2009).

Steps in self-monitoring include:

1. Select a behavior.
   If the behavior is clearly described in specific, observable terms, students will be able to more accurately monitor their own behavior.

2. Collect baseline data.
   Monitor the frequency, duration and/or intensity of the behavior for at least three days before the student begins to self-monitor. The baseline data is useful when determining a reasonable standard for improvement. Comparing baseline data to the students' later performance enables both teachers and students to evaluate the effectiveness of the intervention.

3. Discuss the strategy with the student and obtain the student's agreement to participate.
   Teach the student how to self-monitor and graph the results.
   Establish the time period when the student will monitor the behavior. Provide the student with materials for recording behavior and graphing the results, and teach the student how to use them.
   Provide cues as needed to help the student remember to self-monitor.

4. Implement the agreed-upon plan.

4. Monitor the student's progress.
   Spend time discussing the graph with the student. Celebrate improvement, and help students develop strategies to facilitate further growth.

Guide suggests, "Allow students to chart their progress and to set goals for improvement" (Gersten et al., 2009, p. 46). Students can monitor a variety of their own behaviors, such as attention, participation, and amount of time on task, or aspects of the academic performance, such as accuracy or rate.

While successful students typically monitor their performance intuitively, students who struggle with mathematics frequently lack metacognitive awareness. Therefore, they may need to be explicitly taught to monitor their own behavior. Figure 4.2 provides a summary of self-monitoring and goal-setting. An instructional module describing self-monitoring and goal-setting, with more detailed explanations and multiple examples, can be obtained by accessing Vanderbilt University's IRIS module, "SOS: Helping Students Become Independent Learners" (http://iris.peabody.vanderbilt.edu/sr/chalcycle.htm).

Effective Use of Praise

In recent years, praise has been the subject of hot debate among educators. Teachers hear a great deal of contradictory advice about whether to praise or how to praise their students. As sometimes happens with complex topics, conclusions based upon specific research studies have been overgeneralized, leading to educational recommendations far removed from the actual scientific research. Many teachers have been warned to use praise cautiously because of a mistaken belief that any form of teacher praise undermines student motivation. In fact, while certain kinds of praise do lead to self-defeating behavior, research reveals that "the right kind motivates students to learn" (Dweck, 2008, p. 34).

What is the "right kind" of praise? To answer this question, we must first examine assumptions about human intelligence. At one time, intelligence was viewed as a fixed entity; people thought you were born with a certain amount of ability, and nothing you did could change that innate potential. We now know that the brain is malleable and ability can be cultivated. Learning causes physiological changes in the brain, including larger cortical neurons and more heavily branched dendrites (Wolfe, 2010). In other words, the process of learning makes you smarter. Students who believe that intelligence is fixed and unchangeable tend to seek activities that showcase their intelligence, and they avoid tasks where they might make mistakes, erroneously believing that mistakes reveal a lack of intelligence (Hong, Chiu, Dweck, Lin, & Wan, 1999; Mueller & Dweck, 1998). Studies have shown that students with this fixed view of intelligence are less likely to seek help to correct mistakes, but instead try to hide them. They report that mistakes make them feel dumb and cause them to study less and consider cheating (Hong et al., 1999; Nussbaum & Dweck, 2008). These students believe that individuals who have high ability do not need to expend effort to succeed. They fear that working hard reveals a lack of ability, so they reject tasks that require effort. When they fail, they attribute that failure to lack of ability rather than lack of effort (Blackwell, Trzesniewski, & Dweck, 2007; Marzano et al., 2001). Although students receiving tiered support through RtI have not been specifically targeted for these studies, much of the work has involved students who were identified by teachers as struggling in mathematics. Dweck concludes, "It was the most vulnerable children who were already obsessed with their intelligence and chronically worried about how smart they were" (Dweck, 2008, p. 35).

Successful students, on the other hand, tend to believe that intelligence can be developed. They view mistakes and setbacks as something that can be remedied. When they make a mistake, they study harder or try a new strategy. For these students, effort is a positive attribute integral to the learning process.

How do students develop such different mindsets regarding the role of effort in learning? Researchers investigating this question have found that the kind of praise children receive has a profound influence on their beliefs about the role effort and dedication play in intelligence and achievement. In one study, investigators gave two groups of students various problems and then praised them. One group of children was praised for their intelligence, receiving feedback like "Wow, that's a really good score. You must be smart at this." The other group was praised for effort, hearing, "Wow, that's a really good score. You must have worked really hard" (Kamins & Dweck, 1999; Mueller & Dweck, 1998). The different forms of praise produced dramatically different results. According to one of the researchers:

> The children praised for their intelligence lost their confidence as soon as the problems got more difficult. Now, as a group, they thought they weren't smart. They also lost their enjoyment, and, as a result, their performance plummeted. On the other hand, those praised for effort maintained their confidence, their motivation, and their performance. Actually, their performance improved over time (Dweck, 2008).

In another study, students with learning disabilities in mathematics who received effort-attributional feedback (e.g., "You've been working hard") demonstrated significantly greater academic gains than students who received only performance feedback.

Interventions designed to help students appreciate the importance of effort have improved achievement in mathematics and other areas (for example, see Aronson, Fried, & Good, 2002; Blackwell et al., 2007; Good, Aronson, & Inzlicht, 2003). When teachers praise students' effort and engagement, students often work harder and achieve more (Brophy, 1981; Marzano et al., 2001).

Figure 4.3 Examples of Effective Praise

- You really worked hard on this assignment, and it shows. You stuck with it, and in the end you solved it!
- I like the way you kept working on this, even when it was difficult. Good job!
- You studied for this test, and your improvement shows it!
- You really put forth effort on this. I noticed you got straight to work and stayed focused. That's great!
- This was hard, but you really stuck with it!
- You studied hard. I noticed that you practiced your flash cards and tried the extra problems. That really worked!
- You're really staying focused on this. That's how you'll be successful in the end.
- You paid attention and now you're trying the strategy we discussed. That's great!
- You did well on this! I can tell you worked hard.

Effective praise emphasizes student effort and task-relevant behavior instead of focusing on ability or attributes such as pleasing the teacher or receiving external rewards. Figure 4.3 provides examples of praise statements "that focus on the learning process" (McLeskey et al, 2017).

Rewards

Rewards are increasingly prevalent in today's classrooms. Candy, trinkets, extra credit points, coupons, and pizza parties are frequently offered in an effort to motivate students. Programs in Kansas City and New York City have paid students for attendance and good grades with gift cards and cash (Kumar, 2004).

Research supports the use of incentives to motivate reluctant learners (for example, see Cameron, Bank, & Pierce, 2001; Epstein et al., 2008; Fuchs, Seethaler et al., 2008; Marzano, Pickering, & Pollock, 2001). Providing rewards has led to increased achievement and improved behavior without decreasing intrinsic motivation. In fact, incentive systems that are properly structured can jump-start reluctant learners' motivation. As noted above, cooperative learning programs that included a structured reward system were more effective than those that relied on cooperative learning alone (www.bestevidence.org). Students may initially work to gain the rewards, but once they begin to experience success, intrinsic motivation increases and they begin to experience satisfaction from completing the task successfully. The IES Practice Guide states:

> Tier 2 and Tier 3 interventions should include components that promote student effort *(engagement-contingent rewards)*, persistence *(completion-contingent rewards)*, and achievement *(performance-contingent rewards)*. These components can include praise and rewards. Even a well-designed intervention curriculum may falter without such behavioral supports (Gersten et al., 2009, p. 44).

However, reward systems that are poorly designed or poorly implemented are frequently ineffective and can actually be counterproductive (Deci, 1971; Lepper, Greene, & Nisbett, 1973).

What makes an incentive system effective? To answer that question, we will use a mnemonic created from the letters in the word "INCENTIVE" (Forbringer, 2007). Each letter in the mnemonic represents an element that researchers have found essential to create an effective incentive system. These elements are equally important whether developing an incentive system to use with an individual student or a large group of students. They are appropriate for use during Tier 2 and Tier 3 interventions or in the general classroom. The mnemonic is illustrated in Figure 4.4, which is explained below. The explanation is followed by a sample incentive system that could be effectively used when providing tiered support.

Figure 4.4 Elements of Effective Incentive Systems

I — **Instruction with Incentive**
Is the reward paired with appropriate instructional assistance?

N — **Not Negative!**
Does the incentive system give rewards for positive behavior rather than taking away rewards for inappropriate behavior?

C — **Criteria:**
Is a baseline used to determine what the student must do to earn the reward?

E — **Easy?**
Is the system easy to understand & implement?

N — **Never leave a child with no reason to try!**
Is the system designed so students always have a reason to keep trying?

T — **Timing:**
Is the amount of time students must work to earn the reward realistic for their developmental levels?

I — **Individualized Incentive:**
Is the incentive offered something that will motivate students to put forth the necessary effort?

V — **Verbal Feedback:**
Is verbal feedback provided along with the reward in a way that emphasizes effort ?

E — **Evaluate:**
If the system is not effective, re-evaluate.
Have you followed all eight guidelines for effective incentive systems?

I = Instruction with Incentive

Is the reward paired with appropriate instructional assistance?

To be effective, the incentive must be coupled with appropriate instructional strategies. Tasks must be broken into small steps and carefully sequenced so that a student who expends the necessary effort will be able to complete the task successfully and earn the reward. When an incentive does not produce the desired results, educators sometimes say, "Rewards don't work with this child." Often the error is not with the reward system, but rather that the rewards were not accompanied by the necessary instructional support.

N = Not Negative!

Does the incentive system give rewards for positive behavior rather than taking away rewards for inappropriate behaviors?

Research and best practice both suggest that an incentive system that rewards appropriate behavior is preferable to one that punishes inappropriate behavior (Council for Exceptional Children, 2003; Kampwirth, 1988; Mandella, Nelson, & Marchand-Martella, 2003). Unfortunately, many of the incentive systems recommended online or found in classrooms are negative systems that offer a reward but then withdraw it when the child exhibits inappropriate behavior. For example, one negative system uses stoplights to indicate whether a student can participate in free time or obtain other privileges. Everyone begins the period with a green light, and as long as the student behaves, the light remains green. When the student misbehaves, the light is moved to yellow, and then to red, and the student loses the right to receive the reward. Another commonly used negative system awards points when students are doing well, but takes those points away when problems arise. The reward is dangled before the child, but then taken away if the child misbehaves. The Council for Exceptional Children (2015), the professional organization for professionals who work with individuals with disabilities, specifies that aversive procedures such as response-cost systems may be used, but only as a last resort after more positive interventions have been tried. They are not the best choice for a reward system to be used for an entire class or group of students, and they should not be selected as the motivational system to use in tiered interventions. Response-cost systems may be especially counterproductive with minority students. Research indicates that students from Arab, Asian, and Hispanic cultures respond more positively to quiet, private feedback than to more public correction such as writing names on the board or posting stoplights (Cheng, 1998; Lockwood & Secada, 1999; Walqui, 2000).

Instead of withdrawing rewards when students misbehave, effective incentive systems recognize students when they are engaged in productive behaviors. For example, students may receive bonus points, signatures, or smiley faces when they exhibit desirable behaviors such as beginning work promptly, attending to the task, or completing assignments. These tokens can be accumulated and later exchanged for rewards. Attending to students' appropriate behavior is an evidence-based practice that has been shown to increase task-relevant behavior; in fact, studies of effective classrooms have shown that effective teachers provide four times more attention to students when they are behaving appropriately than the amount of attention given to inappropriate behavior (Majeika, 2020). When teachers use a negative system that involves response-cost, they are forced to attend to inappropriate behaviors. Focusing attention on the child's problematic behavior can foster a negative self-concept and also reinforce inappropriate behaviors in students who find negative attention rewarding. In summary, research findings suggest that incentive systems used during interventions should be positive systems structured to reward productive behavior rather than negative systems that take away rewards for inappropriate behavior.

C = Criteria

Is a baseline used to determine what the student must do to earn the reward?

Incentive systems are most effective when they allow students to experience initial success quickly. Students must believe that they can earn the reward with a reasonable amount of effort. If they perceive the criteria as too difficult, they will rapidly lose interest. Too often students initially respond enthusiastically to a reward system, but become discouraged when their hard work does not produce immediate results. Just as a trainer working with an aspiring pole-vaulter will first set the bar low, then gradually raise it as the young athlete becomes more skillful, so too the criteria for an incentive system should be initially set low and gradually raised as students experience success. To identify effective criteria, instructors should first collect baseline data on the target behavior and use that baseline to determine how well the student must perform in order to earn a reward. For example, if baseline data show that a student typically remains on task for no more than five minutes at a time, it would be unrealistic to require her to remain on task for the whole period in order to earn a reward. Instead, an excellent first step would be to offer a reward if she remains on task for slightly more than five minutes, perhaps six or seven minutes. Students who experience initial success are likely to respond enthusiastically to the incentive system and continue to progress. Just as with the pole-vaulter, the bar can gradually be raised as students gain proficiency. Teachers who shape behavior by rewarding small improvements usually see greater success than those who require more rapid change.

E = Easy

Is the system easy to understand and implement?

An incentive system should improve behavior and increase academic engaged time, not make life more difficult. If explaining or implementing the system requires the teacher to interrupt a lesson in order to record student behavior or to pass out rewards, it can interfere with learning, and the teacher may be tempted to abandon the system. Poorly designed systems can actually create behavior problems. For example, in some systems that focus on negative behavior, when a student is caught misbehaving, the teacher directs him to give back a token, move his name card to the "unsatisfactory" column on the board, or demonstrate failure in some other public way. When this happens, many students will express their embarrassment by becoming increasingly oppositional. A child with a behavior disorder may retaliate by throwing tokens across the room, ripping things off the wall, or otherwise disrupting the classroom. Any system that takes too much time to implement or creates behavior problems in the classroom is not a useful system. The best systems are simple and easy to understand and implement.

N = Never Leave a Child with No Reason to Try!

Is the system designed so students always have a reason to keep trying?

The function of an incentive system is to motivate the most reluctant learners. If a student knows she has already lost all chance of earning the reward before the end of the period, what motivation will she have to keep trying? Conversely, if she realizes part way through the period that she has already done enough to earn the reward, why should she continue working? For example, if the teacher tells a student that she must complete seven out of ten problems correctly in order to earn the reward, then once she has accurately completed seven problems there is little motivation to work carefully on the remaining three. On the other hand, once she has four mistakes, she knows she has no chance of earning the reward and so may give up rather than keep trying.

The best systems allow students to earn a small reward for expending some effort, but a greater reward for expending greater effort. They are designed so the student will always have a reason to keep trying.

T = Time

Is the amount of time students must work to earn the reward realistic for their developmental level?

People work harder when they believe the reward they seek is almost won. Since children's sense of time differs from that of adults, students may become discouraged when expected to work for a time span that seems perfectly reasonable to adults. A reward that the student thinks cannot be attained until the distant future is less motivating than a reward the student believes can be quickly earned. Students may be given frequent points or tokens to show their progress toward earning the reward, but the reward itself should also be offered within a time period that they understand. Canter and Canter (2001) suggest that students in kindergarten and first grade should be able to earn their reward the same day; those in second and third grade may be able to work for two days to a week to attain the reward. Students in fourth through sixth grade should be able to work for one week, and students in grades seven through twelve can work for up to two weeks for a reward. However, students within any given grade differ in their maturity levels. Usually the least mature students in the class are the ones who need the incentive system the most. When deciding how long to ask students to work for a reward, consider the developmental level of the least mature students in the group.

I = Individualized Incentive

Is the incentive offered something that will motivate students to put forth the necessary effort?

If the incentive system is going to work, students must want the reward enough to work hard for it. Just because teachers or parents think a particular item or activity should be rewarding does not mean students will perceive it as rewarding (Cooper et al, 2007; Shea, Bauer & Walker, 2012; Smith & Rivera, 1993). Children differ, and not all children will be equally motivated by any given incentive. The list of potential rewards is almost limitless. Material items include food, trinkets, school supplies, or art supplies. Privileges and social rewards such as extra recess, social time with friends, selecting the assignment, using preferred art materials, being line leader, skipping a problem or assignment, or getting a positive note home can be extremely effective. Rewards can also include teacher attention or bonus points for improvement. To make sure that the incentive offered is something the students in the group find motivating, teachers can watch what the students do during free time and use those items and activities as rewards. They can also ask students what they would like to earn, or use an interest inventory to identify potential rewards. For more information about assessing student preferences, see Cooper et al., 2007. The websites listed in Figure 4.5 suggest rewards that can motivate elementary and secondary students. Although some of these

Figure 4.5 Suggested Rewards

No Salt, No Sugar, & No Money **Acknowledgement Menu: Incentives for Supporting Positive Behaviors** Developed by Effective Educational Practices **www.successfulschools.org**
Illinois PBIS Network's **Non/Low-cost PBIS Reinforcements for Students** **www.slideshare.net/aerobinson1/no-and-lowcost-student-rewards**
Intervention Central Student Rewards-Jackpot Reward Finder https://www.interventioncentral.org/teacher-resources/student-rewards-finder

Figure 4.6 Sample Reward Menus

Reward Menu - Elementary

Piece of candy = 1 point
Sticker = 1 point
Baseball card = 2 points
Markers: Rent for 15 minutes = 1 point
Buy markers = 20 points
Sit in teacher's chair for one period = 3 points
Pass to visit another class for 15 minutes = 5 points
Skip one assignment = 25 points
Earn extra 15 minute recess for class = 30 points
~ Additional rewards will be added throughout the school year~

Reward Menu - Secondary

Piece of candy = 1 point
Borrow pen or pencil = 1 point
First pick of project partner = 3 points
Sit in teacher's chair for one period = 3 points
Bag of chips or candy bar = 5 points
Listen to music during independent work time = 5 points
Earn 5 minutes of talking time for class = 10 points
Pick one problem class can skip on homework = 10 points
Move your desk to a chosen location = 20 points
Free Homework Pass = 25 points
~ Additional rewards will be added throughout the year~

lists include food-based rewards, interventionists should follow school districts' wellness policies when selecting rewards.

Another option is to offer choices for rewards, such as letting students spend reward points on items from a classroom store or select from a list of choices on a reward menu. Students who need immediate gratification can purchase a small item at a small cost, while those able to delay gratification can save their points until they have enough to purchase a more desirable, higher-priced item. Offering choices is especially effective when working with a group of children who might be best motivated by a variety of incentives. See Figure 4.6 for an example of a reward menu.

V = Verbal Feedback

Is verbal feedback provided along with the reward in a way that emphasizes effort?

Whenever incentives are used, they should be paired with social reinforcement such as a smile, a thumbs-up gesture, a pat on the back, or verbal praise. Pairing social reinforcement with incentives develops the student's ability to maintain the desired behavior after the incentive system ends (Shea, Bauer & Walker, 2012).

When students begin to attribute success to their personal effort, research suggests that their achievement will increase (Marzano et al., 2001). Incentive systems are supposed to be temporary interventions designed to help motivate students. Teachers can help students move beyond the need for an incentive system by providing specific verbal feedback along with the earned reward.

E = Evaluate

If the system is not effective, re-evaluate the eight guidelines for effective incentive systems.

A well-designed incentive system will be effective. If the system is not working, the problem is usually with one of the eight elements described above. For example, the criteria may be too high, causing students to become discouraged, or the incentive offered may not be something the students truly desire. When incentive systems adhere to these research-based guidelines, they help students achieve success in tiered interventions. Figure 4.7 provides

Figure 4.7 Example of an Effective Incentive System: The Good Behavior Board Game

The Good Behavior Board game is an incentive system that can be used with a group of students or the entire class. A detailed description of the game, with accompanying blackline masters, is provided by Cipini in *Classroom Management for all Teachers*,(2008). Versions of the game have been studied and found effective with a wide variety of students (Barrish, Saunders, & Wolf, 1969; Carpenter & McKee-Higgins, 1996; harris & Sherman, 1973; Medland & Stachnik, 1972; Saigh & Umar, 1983).

The game requires a large gameboard be posted at the front of the classroom. When students follow classroom rules, they can advance a gamepiece around a board until they reach a treasure box on the board. When they reach the treasure box, everyone in the class participates in a rewarding activity. Before introducing the game, the teacher must first establish clear expectations for behavior, and collect baseline data on the number of rule violations that occur during a10-minute interval. This baseline data is used to determine the criteria that will be required for students to earn an initial reward. For example, if baseline data reveal that the class averages 5 rule infractions every ten minutes, then an appropriate initial expectation would be for the students to show improvement by exhibiting fewer than 5 rule infractions during a 10-minute interval. If the class meets the expectation, they can advance their gamepiece. If they do not meet the expectation, then their game piece does not advance, and they must wait to try again during the next interval. As students become better at controlling their behavior, the behavioral expectations become more rigorous, until eventually students may be required to have zero infractions in order to advance their gamepiece. Gameboards are designed so that students must demonstrate between 5 and 10 appropriate intervals in order to reach a treasure box and earn a reward. Comparing students' baseline behavior to their performance after the intervention has been implemented allows the teacher to evaluate the effectiveness of the intervention.

Reward activities include a variety of privileges such as allowing the class 5 or 10 minutes of social conversation time, minutes to begin working on homework during classtime, time for doodling, listening to music, looking at teen magazines with friends, and other activities that students in the group would find reinforcing.

The Good Behavior Board Game can be evaluated using the mnemonic formed by the letters in the word, "incentive." It contains the elements research studies have determined to be essential in an effective incentive system:

- I = Instruction with Incentive
  This incentive system has been used successfully in a variety of subjects. To be effective, students must be capable of performing the skills required to attain the reward, or be taught how to perform those skills.

- N = Not Negative
  In this game, students move their game piece around the gameboard contingent upon intervals of good behavior. When they reach the treasure box they receive a reward. Students gain rewards for appropriate behavior; they do not lose rewards for inappropriate behavior. If they fail to meet the behavioral goal for a time interval, the gamepiece remains where it was.

Figure 4.7 (Continued)

- C = Criteria
  The Good Behavior Board Game uses a baseline to determine what the students must do to earn the reward. This ensures that the criteria are appropriate for the students in the group.

- E = Easy
  The system is easy to understand and implement It requires minimal preparation time, and game pieces can be advanced quickly with minimal disruption to the academic lesson. When students reach the treasure box they earn the selected prize or activity, but the activity occurs at a time during the day designated by the teacher.

- N = Never leave a child with no reason to try!
  Students always have a reason to keep trying. Even when they fail to advance their game piece, they have another chance during the next 10-minute interval.

- T = Timing
  The length of time students must work is realistic. They can advance their token every 10 minutes. The gameboard can be designed with varying numbers of spaces between treasure boxes, but typically students must advance 5 to 10 spaces in order to earn a reward. If baseline data are used to determine the criteria for advancement, then students should be able to earn a reward every two days to a week. For younger students, the time interval is generally reduced from 10 minutes to 5 minutes, allowing them to earn a reward in a day.

- I = Individualized Incentive
  In this system, the entire class receives the same reward. It is crucial, therefore, that the reward options are carefully selected to include activities and items that will be valued by all members of the group.

- V = Verbal Feedback
  The Good Behavior Board Game does not specify the type of verbal feedback teachers should provide with this incentive system. Teachers will need to use what they know about effective praise to help students focus on their own effort and task-relevant behavior in order to help students develop their own internal locus of control.

- E = Evaluate
  The game directions include suggestions for trouble-shooting that emphasize reviewing the criteria, timing, and selection of appropriate incentives.

Cipani, E. (2008). *Classroom management for all teachers*. Upper Saddle River, NJ: Pearson.

an example of an evidence-based incentive system that has been used to improve behavior and increase achievement in a variety of classroom settings; it would be an excellent choice for increasing motivation among students receiving math interventions. We use the elements described in the mnemonic to critique this incentive system.

Summary

Motivation is critical for success, and many students who require mathematical support lack motivation. Often, they have experienced so much failure that they are no longer interested in putting forth the effort needed to benefit from interventions. Research supports using motivational strategies when working with students who struggle academically.

Several strategies can increase student motivation, including connecting the lesson to real-world applications and students' interests, incorporating active participation and social interaction through games or cooperative learning activities, providing effective praise, teaching students to set goals and monitor their progress, and rewarding task-related behavior and academic performance using incentive systems that follow research-based guidelines. The use of motivational strategies when providing interventions can help all students become mathematically proficient.

5

Explicit Instruction

Students who struggle with mathematics have a variety of learning characteristics that affect their response to instruction. Of the many instructional designs used by teachers, research studies have demonstrated that explicit instruction is the most effective with these students (Gersten et al., 2009; McLeskey et al., 2017). In this chapter, we will first examine important instructional considerations for students who are not making adequate academic progress, then describe the critical elements of explicit instruction that help meet these students' needs, and finally discuss how explicit instruction improves students' motivation.

Instructional Considerations for Struggling Learners

Students who fail to make adequate progress in mathematics frequently have problems with memory and executive functioning (Allsopp et al, 2010; Klesczewski, Brandenburgn, Fischbach et al., 2018; Mabbott & Besanz, 2008; Mazzocro, 2007; Swanson, Jermant & Zheng, 2009). Working memory is the conscious processing of information that enables us to hold small amounts of information in conscious awareness for a short period of time. It allows us to integrate perceptual information with knowledge stored in long-term memory. For example, when a student is asked to mentally add the numbers 3 + 5 + 7, he must process what the question is asking, retain the three numerals in working memory while drawing on his stored knowledge of the process of addition and his knowledge of basic addition facts, and use all that information to obtain a partial sum (3 + 5 = 8) before computing 8 + 7 to obtain the final answer of 15. The average adult can retain about seven items in working memory (plus or minus two) for about eighteen seconds (G. A. Miller, 1956; Wolfe, 2010. You can try this for yourself with a short test. Spend about one second per digit memorizing the following list of seven numbers. Then look away and write them down, in order, from memory:

2 5 1 8 3 4 9

If you have an average memory span you probably remembered all seven, because 7 + 2 is the typical capacity of individuals of age 15 and older (G. A. Miller, 1956). Children have a much more limited capacity, which increases gradually as they mature. The average five-year-old can recall about two items. A seven-year-old can typically retain three items, and a nine-year-old can retain four. By age 11, retention increases to about five items, and by age 13, the average individual can recall about six (Pascual-Leone, 1970).

Individuals who struggle with mathematics generally have less working memory than their normally achieving peers (Chen, Lian, Yang et al., 2017; Fanari & Massidda, 2018; Justicia-Galiano & Martin-Puga, 2017; Lee & Bull, 2016). In today's diverse classrooms, the working memory capacity of students may range from only two to as many as nine items. In addition, the length of time these items can remain in consciousness varies widely among students, creating a challenge for teachers. In order for students with limited working memory to be successful, information must be introduced in small chunks, followed by sufficient practice for the information to be stored in long-term memory before another small chunk of information is introduced. Unfortunately, in many schools, the standard curriculum encourages teachers to present complex problem-solving tasks and to move through content at a brisk pace, under the mistaken assumption that such an approach will offer enrichment for some students while still allowing all students to master the basic information. But look what happens when we ask students to tackle too much information at once. Test your own memory as you did before, but this time, try a list of 11 digits. Spend about one second per digit memorizing the following list of numbers, then look away and write them down, in order, from memory:

8 4 9 7 2 6 5 9 3 1 7

How did you do? If you are like most people, you probably did not do as well on this list as you did on the list of seven items. When we try to overload working memory, the result is similar to what happens when a computer becomes overloaded and all you can do is hit "control-alt-delete." Educators sometimes assume that if they press forward and cover more information, students will retain the basics and only the excess will be forgotten. In reality, when we overload working memory, students typically retain less information than they would have if we had introduced a limited amount of information. In other words, less may be more. Instructional approaches that introduce a small amount of information, then provide time for students to actively process that information before introducing additional information, will be more successful with learners struggling to master mathematics. If an average 11-year-old can retain five items in working memory, then teachers who work with that age group might plan to present a problem that requires students to hold five items in working memory at one time. If the class also contains students with memory deficits, then those students will need simpler problems with more frequent opportunities for review and practice.

In addition to deficits in working memory, students who struggle with mathematics may also show deficits in long-term memory (Geary, 2003). According to the APA dictionary, long-term memory is "a relatively permanent information storage system that enables one to retain, retrieve, and make use of skills and knowledge hours, weeks, or even years after they were originally learned." To be retained, new information is linked to existing information in long-term memory. Some individuals can rapidly form connections and retain new information with minimal rehearsal, while others struggle to connect new information to previous knowledge or need many more repetitions before the new information is successfully stored in long-term memory. In a typical classroom, some students will rapidly make connections on their own and rapidly learn new content, while other students need help forming connections and may need much more practice before the new content is fully retained. For all students, using instructional strategies that help them hook new content to prior knowledge will improve retention. For students with memory deficits, making such connections explicit is even more essential. In addition, students who need more rehearsal time to consolidate learning will benefit from a slower pace and additional practice. Introducing new content too soon disrupts the consolidation of previous learning (Wolfe, 2010).

Individuals who struggle with associations and organizational formats may also have difficulty retrieving previously stored content. This may explain why students with learning disabilities often seem to know something one day but forget it the next. Beginning a lesson by carefully reviewing relevant background information will help these individuals connect new information to prior learning and so improve long-term retention of the information.

Multiple studies have shown that students who struggle to learn mathematics, especially those with learning disabilities, are also less aware of their own cognition than are their normally achieving peers (Bishara & Kaplan, 2018; Montague, 2006; Toraman, Orakci, & Aktan, 2020). They are less likely to recognize whether they have fully understood instruction or to notice whether their answer makes sense, so are less likely to ask for help or clarification when appropriate. As a result, the teacher must take a more active role to make sure these students have fully understood the lesson. For example, the teacher might ask students to explain the lesson in their own words or demonstrate the skill while the teacher watches. Students who struggle with metacognition will also have difficulty selecting an appropriate strategy or following the strategy once selected. Instructional methods that present multiple strategies can be confusing for these students. They benefit from systematic instruction, where they are able to explore one strategy and master it before they are exposed to alternative methods (McLeskey et al., 2017). Multiple-step problems also pose a special challenge, as students may lose track of where they are, resulting in frequent errors (Wong, Harris, Graham, & Butler, 2003). Therefore, students with deficits in self-regulation will benefit from instruction that includes explicit modeling of strategies and multiple opportunities to practice, with support gradually faded as the learners gain competence.

Language-processing difficulties have a profound effect on students' ability to benefit from instruction. Instead of focusing their attention on the mathematics being introduced, students with language deficits must use working memory to process the language, thus reducing the capacity available for mathematical reasoning. For example, these students may have trouble understanding math vocabulary (e.g., factor, exponent, denominator, variable) as they listen to instruction, participate in group discussion, or read and comprehend word problems. Students for whom English is a second language or others with diminished vocabulary will suffer similar problems during instruction. These students benefit from focused instruction in which only a limited amount of information is introduced at one time, vocabulary is explicitly discussed, and teachers frequently check understanding (Powell & Fuchs, 2018).

Finally, students who have done poorly in the past may dread math and begin to think of themselves as mathematical failures (Gersten et al., 2009; Justicia-Galiano & Martin-Puga, 2017). Research indicates that students who have experienced success in the past will persevere even if they can only respond correctly about 75 percent of the time. In contrast, students who do not have a history of success need instruction to be broken down into smaller steps where they can respond correctly 95-99 percent of the time in order to remain engaged (Hunter, 2004). Explicit instruction, which breaks instruction into small steps, can help develop a sense of self-efficacy that will eventually enable these students to attempt more complex and challenging problems.

The Explicit Instruction Lesson

Explicit instruction is the recommended instructional method for students who struggle with mathematics (Gersten et al., 2010; McLeskey et al., 2017; NMAP, 2008). In this section, we describe the key components of explicit instruction: instructional objectives, lesson introduction, review of prerequisite skills and concepts, presentation of new content, guided practice, independent practice, and lesson closure. Figure 5.1 provides a summary of the essential elements of explicit instruction and can be used as a reference as we explore each component.

Instructional Objectives

Explicit instruction begins in the planning stage with a clearly defined instructional objective that describes observable learning outcomes. Having a clear vision of what the students will learn and how they will demonstrate that learning allows teachers to monitor the lesson's effectiveness and to clarify or reteach when necessary.

Effective objectives contain verbs describing observable behaviors that specify how students will demonstrate their understanding. Many published lesson plans contain objectives that state that learners will "understand" the content being presented, but

Figure 5.1 Summary of an Explicit Lesson

Lesson Introduction • **Engage** Make lesson meaningful by providing a real-life example that shows how the skill or concept is used. • **Review Prerequisite Skills & Concepts** Provide opportunity for <u>each</u> student to <u>actively practice</u> any previously mastered skills and concepts necessary for success in this lesson.
Presentation of New Content • Explicitly model strategies and procedures. • Use think-alouds' to describe procedures and the rationale behind them. • Provide multiple examples. • Actively engage students by using multiple modalities and active student participation.
Guided Practice • Have students work with you to perform the same skill you just modeled. Nothing new is added here. • Use a practice format that allows <u>every</u> student to actively demonstrate the skill or concept introduced in the lesson. • Monitor each student and provide immediate feedback. • Have students explain what they are doing and why they are doing it that way. • Provides scaffolded support (i.e. gradually fade prompts as students gain proficiency). • Verify that each student understands the material before assigning independent practice.
Independent Practice • Have students practice the same skill or concept that has just been introduced. (Nothing new is added here.) • Interleave worked examples with problems students work themselves.
Lesson Closure • Have <u>students</u> summarize the main points of the lesson. • Preview next steps.

Figure 5.2 Selecting Effective Verbs

Use Observable Verbs

In the examples below, the verb in the first version of each objective does not describe an observable behavior. The revised version is an observable verb that describes how students will demonstrate their understanding and is more easily evaluated.

- *Problematic verb:* Understand counting to 100
  Revised version: Count out loud to 100 by ones
- *Problematic verb:* Learn how to round whole numbers to the nearest ten
  Revised version: Round whole numbers to the nearest ten
- *Problematic verb:* Know how to subtract fractions with like denominators
  Revised version: Subtract fractions with like denominators

Examples of Observable Verbs		Verbs That Do Not Describe Observable Behaviors	
say write explain draw solve circle copy label define point to	select name type print state construct count out loud describe put in order model	know realize appreciate discover learn realize value feel gain familiarity with	believe improve recognize understand work on think about increase understanding be familiar with

do not specify how that understanding will be demonstrated or evaluated. Because the teacher cannot see inside the student's head and cannot evaluate what the student understands, knows, believes, or realizes, such verbs are best avoided. Objectives specifying that the student will explain, construct, or model the concept can be easily evaluated. For example, an objective that states that the student "will improve understanding of two-digit multiplication" is less precise than an objective that specifies that the student "will correctly multiply two two-digit numbers where regrouping is required." Likewise, an objective that says the student "will gain familiarity with halves, thirds, and fourths" is harder to evaluate than an objective that specifies that the student "will correctly model ½, ⅓, or ¼ of a whole or set." Figure 5.2 provides examples of effective and ineffective verbs.

Effective objectives do not describe instructional activities, but focus on the outcomes of instruction. Learning can occur through a variety of instructional activities, and an effective teacher will switch methods, explain the concept differently or provide a different activity when what was planned is not working. The intended learning outcome is not changed by this adjustment in methods, however. For example, "The students will play a game to review skip counting" is a description of an instructional activity, not a learning outcome. Revising the objective to state "Students will skip count to 100 by 2's, 5's, and 10's" provides a much clearer description of the intended learning outcome. This objective would remain unchanged whether the students played a game, completed a worksheet, or used a computer-based activity. Figure 5.3 provides additional examples of revising objectives to clearly specify what students will learn.

Figure 5.3 Effective Objectives Describe Learning Outcomes

Effective Objectives Describe Learning Outcomes

In the examples below, the first version describes an instructional activity. The revised objective is preferred because it specifies the learning outcome.

- *Problematic objective:* Read word problems with a partner and decide which information is needed to solve the problems
  Revised objective: Given word problems that contain extraneous information, students will identify which information is needed to solve the problem.

- *Problematic objective:* Complete the review sheet on page 32
  Revised objective: Solve single-digit addition problems, sum ≤ 18

- *Problematic objective:* Practice using base ten blocks to model numbers to 1000
  Revised objective: Given a number less than 1000, model it with base ten blocks

Effective instructional objectives describe the behaviors that demonstrate learning and so help teachers monitor student understanding and make necessary adjustments to their methods of instruction without changing the instructional objective.

Lesson Objective

Engage

Our bodies are biologically programmed to attend to and remember information that is meaningful. When students perceive a lesson as having personal relevance, they will be more attentive and efficient learners (Archer & Hughes, 2011; Wolfe, 2010. Therefore, the teacher's first task is to engage the students. Students want to know, "Why do we have to learn this?" In mathematics, regardless of the instructional method being used, making information meaningful and relevant means that new skills and concepts should be introduced in the context of solving real-life problems (NCTM, 2000). For example, students interested in baseball will be more engaged if the process of finding averages is introduced by calculating batting averages; students who enjoy cooking may understand fractions better when they are presented as a problem about measuring ingredients. Beginning a lesson on dividing using two-digit divisors by saying "Today we are going to learn about division" is far less engaging than beginning with a real-life problem like the following:

> Spring break is coming, and our family is going to drive to Florida for vacation. I'm trying to figure out how much this trip will cost, and one big expense will be gasoline. I looked on Google Maps and learned that it's 868 miles from here to our destination. My car can go 28 miles on one gallon of gasoline. Can you help me figure out how many gallons of gas I will use driving from here to Florida?

Unfortunately, most textbooks do not introduce mathematical procedures in the context of meaningful problem-solving. One review of fraction lessons in middle-school textbooks found that less than 10 percent of the lessons presented new fraction concepts in a meaningful context (Hodges, Cady, & Collins, 2008). Therefore, teachers are often forced to develop their own examples to make the content meaningful. When districts adopt new textbooks, selecting materials that introduce new content in the context of solving real-life problems can boost student achievement.

Review of Prerequisite Skills and Concepts

In order to profit from the current lesson, students often must have mastered pre-requisite skills and concepts. For example, before students can learn to regroup in subtraction, they should already be able to subtract without regrouping and to understand place value. Before counting by 5's, students should be able to count by 1's. Before learning to add coins, they need to recognize the coins, know their value, and be able to add. Therefore, when planning a new lesson, it is essential that the teacher make a comprehensive list of any pre-requisite knowledge students would need in order to succeed in this lesson, and systematically check to make sure *each* student in the group has the necessary background knowledge before the new content is introduced. This is the principle of systematic instruction, which has been labeled as a high-leverage practice for students who struggle academically (McLeskey et al, 2017). The objective for an effective lesson will fall within Vygotsky's "zone of proximal development" (Vygotsky, 1978), that small area just beyond the student's current level of performance, but within reach when presented with guided support. If the objective is too advanced and too much new information must be introduced at one time, then the student's working memory will be flooded and little learning will occur. If the students have not mastered these prerequisites, then the missing information should be introduced in a separate lesson before proceeding with the new objectives.

During the lesson introduction, the teacher reviews these previously mastered pre-requisites. This review has two objectives. It allows the teacher to evaluate each child's readiness for the upcoming lesson, and it provides valuable review for the students. As mentioned previously, many students who perform poorly in mathematics have difficulty retrieving previously learned information. Without the review, these students may spend valuable time during the lesson trying to recall pre-requisite skills and so miss critical instructional input.

In explicit instruction, activating prior knowledge does not mean simply asking students, "Do you remember when we worked on this procedure last week?" Students who do not remember will seldom risk embarrassment by admitting their ignorance before their peers. It is also not sufficient to simply ask a couple of students to come to the board to do a sample problem or explain the information. Although the volunteers doing the explaining may have the necessary pre-requisites, this procedure gives the teacher no information about how well the rest of the class has mastered and can recall the pre-requisites. In an explicit instruction lesson, the teacher reviews pre-requisites by providing a task that requires each individual in the group to demonstrate understanding. If the group is small, this review may be accomplished by having students complete a few review problems at their desks. Such a procedure can provide effective review for the students, but it will only provide useful formative assessment data for the teacher if the group is small enough that the teacher is able to monitor each student's work and verify understanding before proceeding. If the group is too large for the teacher to monitor each student's output, then it may be more effective to have students provide a more easily monitored response. For example, if the teacher wants to check students' recall of coin values, she could provide each child with a whiteboard and marker, then hold up a coin and ask students to write the coin's value on their whiteboards and hold the boards up for her to see. Knowledge of numerals could be reviewed by displaying a numeral and asking students to hold up the corresponding number of fingers. Recognition of prime and composite numbers could be evaluated by holding up a number and having students give a thumbs-up if it is a prime number and thumbs-down if it is a composite number. Procedures like these allow every child to actively participate while simultaneously allowing the teacher to gauge their understanding. Review should continue until all students are fluent with the necessary background information.

Presentation of New Content

Students struggling with mathematics benefit when new information is first clearly modeled. This is sometimes referred to as the "I do it" portion of the lesson. When a new skill or strategy is introduced, the teacher needs to explicitly model the process. The teacher demonstrates the procedure while simultaneously describing it. For example, if the students are learning to write the numeral "2," the teacher would first demonstrate how to form the numeral while simultaneously verbalizing the process. Pairing the visual demonstration with a verbal description of the process will increase retention by modeling self-talk students can use when executing the process themselves. While demonstrating how to form a "2," the teacher might say, "Start near the top, curve up to the top line, curve down to the bottom line, then straight across. That's a 2. Curve up, curve down to the bottom, then straight across."

Describing your thoughts and actions as you perform the skill is often referred to as "think-aloud." The words used in the description should be simple, clear, concise, and consistent. Since many students who have difficulty learning mathematics also have language deficits, using simple language will allow students to focus on understanding the information rather than being distracted by difficult vocabulary. Modeling multiple examples and consistently using the same words each time provides repetition that facilitates retention and helps students internalize the procedure.

In the following example, a teacher uses think-aloud to model comparison notation. Before introducing the comparison symbol, the teacher would review by having students practice identifying the larger or smaller of two numbers, which is a pre-requisite for this skill. To make the symbol more meaningful to the students, she might compare it to the mouth of a hungry alligator that wants to devour the largest number it can find. Then the teacher models how to insert the symbol between two numbers. She puts the problem "8 __ 5" on the board and uses think-aloud to share her thoughts as she inserts the comparison symbol between the two numbers.

> Let's see. First, I need to decide which number is bigger. I know that 8 is bigger than 5. We said that the hungry alligator wants to eat the biggest number it can find. That means the open mouth needs to point toward the 8. So I'm going to draw my symbol with the open part facing the 8, like this. *(Inserts the symbol to show that 8 > 5.)* Now the mouth is ready to eat the bigger, or greater number. I read the expression from left to right: 8 is greater than 5.

When introducing more complex procedures, teachers should provide step-by-step models, thinking aloud as they model each step (Archer & Hughesm, 2011; Gersten et al., 2009, McLeskey et al., 2017). Students who have deficits in metacognition have difficultly selecting and executing appropriate strategies. When presented with multiple-step problems, these students need clear, unambiguous models of each step in the complex process. The verbalization not only describes *what* to do, but also provides insight into *why* to do it.

In Figure 5.4, a teacher uses the think-aloud strategy to model an explicit strategy for subtracting two-digit numbers when regrouping is required. Pre-requisites for this skill include regrouping in addition, computing basic subtraction facts, and understanding place value (decomposing numbers, expanded notation, and "making trades" to exchange 10 ones for 1 ten). Students should have demonstrated mastery of these prerequisites during previous instruction. During the lesson introduction, the teacher would review these pre-requisites

Figure 5.4 Modeling an Explicit Strategy for Regrouping in Subtraction

Steps for Regrouping in Subtraction

1. Show how many I have (the total).
2. ONES Column:
    - Decide: Need to regroup?
    - TO REGROUP:
        - Make a trade.
        - Record.
    - Subtract ones & record.
3. TENS Column: Subtract & record.
4. Check.

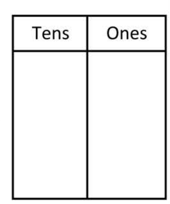
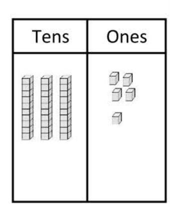
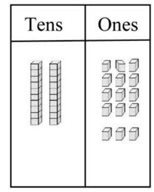
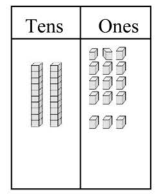
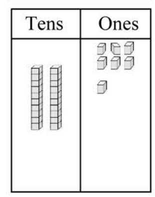
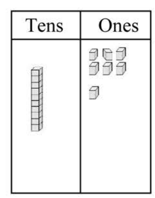
The teacher says:	The teacher shows:
Here's our problem. We have 35 science books in the room, and we need to keep 18 of them here so each of you can have a book. We want to know how many books we have left that we could let Mrs. Rivera use.	35 –18
I'm going to follow these steps to solve the problem. *(Points to the steps listed above.)* Step 1 says, "Show how many I have." Let's see. We have 35, so I need to use my base ten blocks to show 35. First I'm going to make a place to put my tens and ones. I'll draw lines and label the columns 'ones' and 'tens.' *Draws a place value chart.*	35 –18 
OK. Now I need to lay out 35. I can use my base ten blocks to show 35 with 3 rods and 5 units. That's 3 tens in the tens column, and 5 ones in the ones column. *Lays out 35 blocks.* Let's see. Have I finished step 1? It says, "Show how many I have." I've done that, so I'm going to check off step 1. *Puts a check next to #1 in the steps listed above.* I don't lay out 18 blocks, because the bottom number in a subtraction number tells me how many I will need to take away from the total.	35 –18 
Step 2 says, "ONES column. Decide: Do I need to regroup?" Alright, I'm going to subtract the units in my ones column first. I have 5 ones, but I need to take away 8 ones. I don't have enough ones to do that, so	

Figure 5.4 (Continued)

I guess I need to regroup. It says, "To regroup, make a trade." I remember when we played "Making Trades" that I can break down a ten and change it to ten ones, without changing the total amount. I'm going to do that. I'm going to take one of my tens and break it apart. Now I have ten more in the ones column. I had 5, and I've added 10 more, so now I have 15 ones. I broke down a ten and moved it to the ones. I've done that, so I can check that off. *Checks off "REGROUP" in the steps listed above.*	**35** **−18** 
Next, it says, "Record." I used to have 3 tens, but I regrouped and now I only have 2 tens, so I'll cross of the 3 and change it to say I have 2 tens. I used to have 5 ones, but now I have 15 ones, so I need to make my problem say that. Alright. I've completed that step, so I can check it off. *Checks off "Record."*	2 1 **~~3~~5** **−18** 
Now it says, "Subtract ones, and record." O.K. I have 15 ones and I'm supposed to take away 8. I'm going to do that. Now I've got *(counting)* 1, 2, 3, 4, 5, 6, 7 left. That makes sense because I know that 15 – 8 = 7. I need to record the answer, so I'll write 7 in the ones column. Now I can check that off. *Checks off "Subtract ones & record."*.	2 1 **~~3~~5** **−18** **7** 
Step 3. "TENS Column: Subtract and record." I have 2 tens and I need to subtract 1 of them. I'm going to take one bundle of tens from the tens column. That means I have 1 ten left, so I'll write '1' in the tens column. I can check off step 3, and I'm done. *Checks off Step #4 above.*	2 1 **~~3~~5** **−18** **17** 
Step 4. Check. Let's look at our problem. Let's count and see if the blocks match what we have written. I have one ten and 1, 2, 3, 4, 5, 6, 7 ones. That's 17. It matches! Ok, we have solved our problem! Right now our class has 35 science books and we're using 18 of them. We figured out that we have 17 left over that we can lend to Mrs. Rivera's class.	

and engage the students in the new skill by providing a real-life application of regrouping such as the following problem:

> Mrs. Rivera came to me this morning and asked to borrow the science books on our shelf. I told her we'll be using some of them, but I would see how many extras we have available. We have 35 books on the shelf. There are 18 students in this class, so we need to keep 18. How many books do we have left over that we could let Mrs. Rivera use?

Note that although the example begins with a word problem, the focus of this lesson is on teaching students the standard algorithm for regrouping in subtraction. The word problem provides a context for teaching the algorithm. The process for solving word problems would be taught in a separate lesson, because students with memory deficits or problems with executive control often become overwhelmed when lessons introduce too much information at one time.

Explicitly modeling strategies does not mean teaching students rote procedures without developing conceptual understanding. In Chapter 3, we discussed the importance of using the concrete-pictorial-abstract (CPA) continuum to help students understand mathematical concepts. Students should first have hands-on experience using concrete models and then progress to pictures, tallies, or other types of illustration before being asked to execute a procedure using only abstract words and symbols. Exploring concrete models brings meaning to the abstract concepts. Students are not practicing rote procedures, but rather developing conceptual understanding of the rationale underlying these procedures. Explicit modeling is integrated into each step of the CPA continuum to help students understand and execute procedures. Using the CPA continuum to develop conceptual understanding will be discussed in more detail in Chapter 6. In addition, it is important to have students discuss what they are doing, and why they are doing it that way. Asking students to verbalize their reasoning is a powerful way to deepen understanding.

During the modeling portion of the lesson, the teacher first demonstrates how good problem-solvers approach a problem using multiple clear, unambiguous examples (Gersten et al., 2009, p. 22). Providing multiple examples allows the student to see a strategy applied in a variety of contexts and increases the probability that students will remember and be able to apply the strategy themselves when presented with similar problems in the future. This important feature should be included as a criterion when selecting intervention materials (Gersten et al., 2009).

After effective examples are provided, it is also helpful to model examples that include errors. Modeling incorrect use of a strategy allows the teacher to demonstrate self-monitoring, a skill that many learners with disabilities lack. The teacher can show students how to recognize whether their answer makes sense and how to correct errors. However, it is best if in the initial presentation students model the correct procedure and only the teacher models making mistakes. There will be natural opportunities for students to practice self-correction during instruction.

Modeling strategies does not mean that the teacher does all the talking and thinking while students are passive recipients. Students learn more when they actively participate. In Figure 5.4, the teacher missed multiple opportunities to involve students. Place value was identified as a pre-requisite skill, so the teacher might have asked students to point out the ones column or help break down a ten. Knowledge of basic subtraction facts was also a pre-requisite, so the teacher could have asked students to compute $15 - 8 = 7$ when completing step 3. During subsequent examples, the teacher could involve the students in following the strategy steps. For example, the teacher might ask the students, "What is step 1?" When students respond, "Show how many I have," the teacher could then ask, "What does that mean for this problem? How many do I have? How will I show it?" Involving students in the process helps maintain their attention, and their active participation increases learning.

Guided Practice

Guided practice has two important functions. Students practice under the teacher's guidance and receive corrective feedback that enables them to correct any misunderstandings, and teachers receive formative feedback that enables them to evaluate *each* student's understanding. The skill practiced should be the same skill that has just been modeled. It is not an extension activity; students are not asked to go beyond what was modeled, but rather to do themselves what the teacher has modeled.

In many mathematics classrooms, teachers demonstrate a few examples on the board and then students are given a worksheet or textbook assignment and asked to practice the skill on their own. This procedure is sometimes justified by saying, "Practice makes perfect." But the simple act of practicing something repeatedly does not automatically lead to perfection. Practice makes permanent. If we want to perform perfectly, we need to practice perfectly. In other words, "Perfect practice makes perfect." If students are asked to practice a skill independently before they can perform it correctly, they will practice mistakes, and those mistakes can be very hard to unlearn. Anyone who has tried to help a student unlearn previous misconceptions appreciates how frustrating and time-consuming it can be for both teacher and student. In addition to having a negative impact on student learning, moving too quickly to independent practice can also negatively impact motivation. Students who have experienced failure in the past may resist activities where they anticipate another failure. They may become off-task or disruptive in an effort to avoid the task, or they may rush through an assignment because their past experience has shown that, even if they work hard, they are unlikely to experience success.

During guided practice, the teacher provides scaffolded support as needed. The teacher may provide visual prompts, such as posters or cue cards listing the steps to follow, or remind students of a critical aspect of the problem. Support also includes verbally prompting the student on what to do next or asking the student process questions. Process questions require students to describe the process they are using, explain their reasoning, create an example, or prove that their answer makes sense. Students think aloud just as the teacher did during the modeling phase of the lesson. The *What Works Practice Guide* recommends:

> During guided practice, the teacher should ask students to communicate the strategies they are using to complete each step of the process and provide reasons for their decisions. In addition, the panel recommends that teachers ask students to explain their solutions. Note that not only interventionists—but fellow students—can and should communicate how they think through solving problems to the interventionist and the rest of the group. This can facilitate the development of a shared language for talking about mathematical problem-solving (Gersten et al., 2009, p. 23).

The process of articulating their reasoning helps students consolidate understanding. In chapter 2, we provided examples of process questions that teachers can use to develop and evaluate student understanding (see Figure 2.4).

Prompts are gradually faded during the guided practice portion of the lesson. How quickly the teacher fades support depends on the complexity of the task and the success of student performance. Guided practice should continue until students are able to perform successfully on their own. At the beginning of guided practice, the teacher might present a new example and ask the students to give step-by-step directions as they guide the teacher through solving the problem. The teacher would provide prompts as needed, gradually fading these prompts as the students demonstrate competence. Then the teacher would ask students to demonstrate their ability to perform the skill independently without prompts. The IES Practice Guide recommends, "For students to become proficient in performing mathematical processes, explicit instruction should include scaffolded practice, where the teacher plays an active role and gradually transfers the work to the students" (Gersten et al., 2009, p. 23).

In effective guided practice, every student actively responds in a way that allows the teacher to evaluate each student's level of understanding. For the lesson presented in Figure 5.4, the teacher might ask students to solve a few problems using individual whiteboards and then hold up their answers while the teacher checks for understanding. If there are only a few students in the class, the teacher could monitor each of them as they work on problems at their desks, but with a larger group, it would become difficult for a teacher using this procedure to effectively monitor each student's understanding. Small-group activities can provide effective guided practice if they are structured to ensure that every student actively participates and receives scaffolded support and corrective feedback in a timely manner. See the online materials for examples of some guided practice formats.

Independent Practice

In explicit instruction, students practice independently only after they can perform successfully without prompts. The purpose of independent practice is to provide the student with sufficient repetition for new learning to be effectively stored in long-term memory. Students who struggle to learn new content often have deficits in working memory (Chen, Lian, Yang et al., 2017; Fanari & Massidda, 2018; Justicia-Galiano & Martin-Puga, 2017; Lee & Bull, 2016) and need extensive practice opportunities before they successfully consolidate new learning. Core instruction typically moves at a faster pace and provides less practice than these students need to become proficient.

Massed Practice

Immediately after instruction students need to solve multiple examples of similar problems, using the same strategies modeled during instruction. Practice periods are "massed," meaning that several practice periods are scheduled close together. For example, after successful guided practice, students might have several problems to complete independently before the end of the math period. They might do more examples for homework, and practice again during class the next day. Additional examples would be reviewed at the beginning of class the following day.

Problems presented during initial independent practice should include worked example solutions as well as problems for students to solve independently. Research studies have shown that students learn more when examples of worked-out problems alternate with problems students must solve independently (Agarwal & Agostinelli, 20020; Chen, Retnowati & Kalyuga, 2020). These studies have demonstrated that asking students to solve eight practice problems independently results in less learning than if the eight problems include four completed examples that show solution steps, alternating with four problems students must solve on their own. Even though students actually solve fewer problems in the second scenario, they are more likely to solve those problems correctly and perform better on similar problems in the future. When giving students initial independent practice assignments, learning will increase if teachers provide a worked-out solution for every other problem. See Figure 5.5 for an illustration of interleaving worked example solutions and problem-solving exercises.

Figure 5.5 Interleaving Example Solutions and Problem-Solving Exercises

Multiplication:

Use partial products to solve these problems.

Study each step in the examples given to help you solve the next problem on your own.

12	34	45	17
x 13	x 12	x 23	x 18
6		15	
30		120	
20		100	
100		800	
156		1035	

Unfortunately, most materials currently available do not provide sufficient worked-out examples for teachers to use. Until resources incorporate these research findings, teachers or teams of teachers will need to develop their own worked examples for the content they teach.

Problems presented during massed practice should also include discrimination examples that do not require use of the new skill or strategy. For example, the teacher in Figure 5.4 was teaching students to regroup in subtraction. During independent practice the teacher would include examples that require regrouping, but should also include examples that do not require regrouping. Inserting discrimination items prevents students from automatically and thoughtlessly regrouping in every problem they encounter. When students think critically about what they are doing and why they are doing it, they are better able to apply their knowledge appropriately in the future.

Distributed Practice

Once students have mastered the new content, the practice schedule moves from massed to distributed. Distributed practice involves providing short practice periods, with longer and longer time intervals between reviews. Such review increases long-term retention and also develops students' ability to retrieve previously stored content. Current mathematical resources often fall short in this area, too. In many basal programs, students focus on one unit, then another, without the distributed practice needed for long-term retention. For example, students might complete a unit on multiplication, then move on to fractions, then to measurement, without sufficient opportunity to review the content covered in previous units. Students with memory deficits are especially harmed by insufficient distributed practice. Sometimes teachers work hard to help their students master concepts early in the year, only to find they have forgotten the material later. Frequently, this occurs because students did not receive sufficient distributed practice opportunities. Providing cumulative review helps students maintain knowledge over time.

Lesson Closure

How much of the explicit lesson is completed during one instructional period will vary depending on the students, the complexity of the information, and the length of the period. Sometimes you may get through the introduction, presentation of new content, and guided practice before the period ends and even have time to begin independent practice. Other days you may only get halfway through the guided practice portion of the lesson before it is time to close. However far the lesson has progressed by the end of the instructional period, it is important to summarize the learning that has occurred that day. Summarizing crystallizes key points in the students' minds. Rather than telling the students what they learned, the teacher asks the students to create the summary. The processes of selecting and articulating the important information deepens understanding and increases retention. These summaries can take many forms. The teacher might call on individual students to share their responses, ask students to share their ideas in a think-pair-share format (Kagan, 1994), or have students write or draw their ideas in a daily journal. For the subtraction lesson described in Figure 5.4, the teacher might end the lesson by saying the following:

> Today we learned how to regroup in subtraction. What does it mean to regroup? *Calls on student to answer*. How do you know you need to regroup when you're subtracting? *Calls on student to answer*. On your whiteboards, write an example of a subtraction problem that requires regrouping. *Monitors responses, then calls on a student to explain her*

response. We don't always regroup every time we subtract. What kind of problem can be solved without regrouping? *Calls on a student to answer*. Think to yourself the steps we follow when we regroup, and then turn and share your ideas with your partner. *Monitors the think-pair-share, then calls on one pair to share their response with the whole class*. Now think of a real-life situation where you might need to regroup. Share your idea with your partner. *Monitors the think-pair-share, then asks one or two students to share their examples with the whole class*.

Notice that the summary is not merely a statement of the topic that was covered (regrouping in subtraction), but requires students to summarize the main points of the lesson.

How Explicit Instruction Improves Motivation

Explicit instruction has been shown to improve outcomes and help struggling students experience mastery. The following specific aspects of explicit instruction facilitate success and therefore lead to higher motivation.

1. During the lesson introduction, activating necessary background knowledge and pre-requisite skills prepares learners to use that information when new content is introduced. Students are less likely to become frustrated because they have forgotten previously mastered content and are more likely to approach the lesson with a positive "can-do" attitude.
2. Introducing information in small, carefully sequenced steps minimizes frustration and allows individuals with memory deficits and cognitive processing problems to experience success.
3. Providing scaffolded support increases success. When teacher support is faded gradually, students gain confidence and are more willing to tackle similar problems on their own in the future.
4. Guided practice allows the teacher to verify student understanding before asking students to practice independently. The teacher can correct misunderstandings quickly, which helps ensure that students will be able to work successfully on their own. When students are performing successfully, the teacher can provide positive feedback, thus boosting students' confidence.
5. Frequent review increases the probability that students will retain previously mastered content. The more clearly students remember what they learn, the more confidence they will feel when approaching future mathematical tasks.

Since explicit instruction has produced strong positive results among students who have had mathematical difficulties, the IES practice guide recommends it be used in all tiered interventions (Gersten et al., 2009) It has also been identified as the evidence-based practice for use with all students with academic disabilities (McLeskey et al., 2017).

Summary

High-quality research studies have consistently demonstrated that explicit instruction can improve the mathematical achievement of students who struggle with mathematics, and it is the recommended instructional method for students receiving mathematical interventions. Research suggests that students who fail to make adequate progress in mathematics often have problems with working memory, long-term memory, and executive functioning.

The typical inquiry-based lesson favored by many mathematics educators presents complex tasks that overload the memory capacity of these students and challenge their metacognitive abilities. With explicit instruction, information is presented in smaller chunks and is therefore more accessible for students with memory deficits.

The key components of explicit instruction include lesson introduction, review of pre-requisite skills and concepts, presentation of new content, guided practice, independent practice, and lesson closure, which were summarized in Figure 5.1 at the beginning of the chapter. In the online materials, we provide sample lesson plans that follow the explicit instruction model, and a form that can be used when designing an explicit instruction lesson. In the following chapters, we discuss how to use concrete and pictorial representation to develop understanding, and we provide additional examples of explicit instruction to support learners who struggle with mathematics.

6
Concrete and Visual Representation

Students who are successful in mathematics have a rich sense of what numbers mean and can engage in quantitative reasoning. If the teacher says the word "twelve," an individual with a robust sense of 12 will easily envision 12 objects, 12 tally marks, a dozen eggs arranged in two rows of six, a 3x4 array, a group of 10 + 2, the numeral "12," and so on. All of these are different ways of representing 12. The ability to represent mathematical quantities in multiple ways is a critical component of quantitative reasoning. Representation allows students to organize mathematical information, describe mathematical relationships, and communicate mathematical ideas to others. According to the National Council of Teachers of Mathematics (NCTM), "Representing ideas and connecting the representations to mathematics lies at the heart of understanding mathematics" (NCTM 2000, p. 136). To highlight the essential role representation plays in effective mathematics instruction, NCTM developed the Representations Standard, which states:

Instructional programs from pre-kindergarten through grade 12 should enable all students to:

- Create and use representations to organize, record, and communicate mathematical ideas;
- Select, apply, and translate among mathematical representations to solve problems;
- Use representations to model and interpret physical, social, and mathematical phenomena.

NCTM 2000, p. 67

Representation can take a variety of forms. Students can use concrete objects to demonstrate mathematical concepts, as when they use toy blocks to model an addition problem or cut a pizza into eight equal pieces to illustrate fractional parts of a whole. They can draw pictures of those same objects, or use tally marks or graphs to record quantity. Mathematical ideas can also be represented in words and symbols. The process of representing their ideas helps students construct meaning, as well as organize and clarify their thinking. Linking various representations of the same mathematical concept or procedure deepens a student's understanding of mathematics.

Research on the Concrete-Pictorial-Abstract (CPA) Sequence

Conceptual understanding of quantity follows a developmental sequence. It begins at the concrete level when students physically act out a math problem, or use manipulatives such as blocks, toothpicks, fraction pieces, or other three-dimensional objects to model a mathematical relationship. Manipulatives include any objects that students can physically manipulate and that provide concrete representation of a mathematical idea. However, just because a student can manipulate an object does not mean that it can be effectively used to model a mathematical concept. For example, students can manipulate coins, but coins are not the best manipulatives to introduce decimal tenths and hundredths, because coins do not provide a concrete model of their relative values. Pennies and nickels are larger than dimes, yet their values of $.01 and $.05 are both less than the value of the dime. The manipulatives that we use to represent mathematical concepts should provide a clear visual model of the concept. Base-ten blocks are more effective for modeling the relative values of decimal tenths and hundredths, because they clearly show the value of the different decimals. If a flat (10x10 square block) represents one whole, then a rod (1x10 block) represents .10 and a unit (1x1 block) shows the value of .01. Coins provide a real-life example of why understanding decimals is important, and their use in lessons is encouraged. However, students need to experience manipulatives like base-ten blocks that provide a concrete illustration of the relative sizes of different decimals during initial instruction. Playing cards represent another example of three-dimensional objects that students can manipulate, but which do not concretely model mathematical concepts. Cards are useful to increase student engagement, but the cards contain pictorial and abstract representations of numerical values, not concrete representation of the relative values of the numerals. Card games provide effective, engaging practice materials, but the manipulatives students use during initial instruction should provide concrete representation that enables them to create a clear model of the concept. Research has long established that, when students have the opportunity to use concrete materials, their mental representations are more precise and comprehensive, they have an increased understanding of mathematical ideas, they are better able to apply them to real-life situations, and they often demonstrate increased motivation and on-task behavior (Harrison & Harrison, 1984; Suydam & Higgins, 1977).

As their understanding deepens, students progress to using pictorial representations such as drawings, tally marks, diagrams, graphs, charts, and tables or other two-dimensional illustrations to model mathematical concepts and procedures. With practice, they learn to connect these two and three-dimensional representations to abstract words and symbols until finally the words and symbols are meaningful by themselves and students are able to efficiently represent mathematical concepts and procedures using just the abstract symbols. In Chapter 5, we provided an example of explicitly linking concrete, pictorial, and abstract representations when teaching the standard algorithm for regrouping in subtraction (see Fig. 5.4). Figure 6.1 shows how multiplication could be represented using concrete, pictorial, and abstract representation.

When mathematical words and symbols are firmly rooted in concrete experiences, students find them meaningful. When students lack a foundation in concrete and visual representation, their attempt to perform symbolic operations often becomes a rote execution of meaningless procedures.

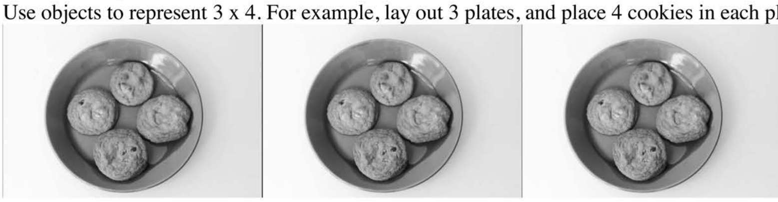
The progression of understanding is referred to as the concrete-representational-abstract (CRA) continuum, or the Concrete-Pictorial-Abstract (CPA) continuum (Sousa, 2007; Witzel, 2005). Based on the work of Jerome Bruner in the 1960s (Bruner, 1960), the CPA continuum has been supported by mathematics educators for decades. NCTM recommends incorporating

Figure 6.1 The CPA Continuum

The representations below are designed to use when teaching the following Common Core Standard:

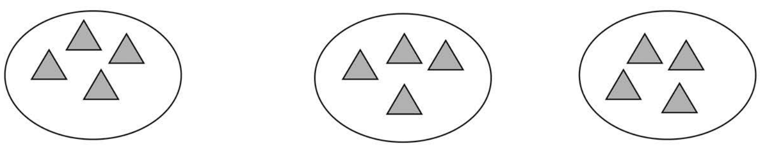
CC.3.OA.1.Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. ***For example, describe a context in which a total number of objects can be expressed as 5 × 7.***

Concrete Representation:
Use objects to represent 3 x 4. For example, lay out 3 plates, and place 4 cookies in each plate.



Pictorial Representation:

Use a picture to show 3 x 4. For example, draw 3 circles to represent the 3 plates, then draw 4 cookies on each plate. The student might use circles, triangles, dots, or other shapes to represent the cookies.



Abstract Representation:

Represent the problem in <u>words</u>.
There were 3 children at my party. I gave each child 4 cookies. In all, how many cookies did I give away?

Represent the problem using numbers and symbols: **3 x 4 = 12**

the CPA continuum in core instructional materials at every grade level (NCTM, 1989, 2000), and it is a key component of the Common Core State Standards for Mathematics (2010). It is an evidence-based practice for all grade levels that is supported by a large body of research (see, for example, Bouck, Park & Nickell, 2017; Fuchs, Powell et al., 2008; Gibbs, Hinton & Flores, 2017; Peltier & Vannest, 2017; Sealander, Johnson, Lockwood & Medina, 2012; Witzel, Mercer & Miller, 2003; Woodward, 2006). Following the CPA continuum is especially critical for students who struggle with mathematics. After an exhaustive review of the research, authors of the IES Practice Guide (2009) included it as one of their eight recommendations for assisting students who struggle with mathematics: "Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representation of mathematical ideas" (Gersten et al., 2009, p. 6).

Pacing CPA instruction is an important consideration when designing lessons for students who have traditionally struggled with mathematics. Individuals with learning disabilities have been found to require about three lessons at the concrete level, each consisting of approximately 20 problems, before they are ready to fade concrete support, and then three more lessons at the pictorial level before they have developed conceptual understanding and are ready to work solely with abstract words and symbols (Hudson & Miller, 2006).

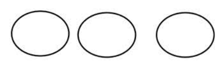
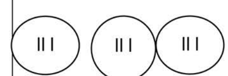
To progress from one level to another level within the CPA sequence, Hudson and Miller suggest that students should be able to complete ten independent practice problems with at least 80 percent accuracy. These findings are consistent across a range of mathematical content from basic number sense to algebra (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Harris, Miller, & Mercer, 1995; Hudson, Peterson, Mercer, & McLeod, 1988; Mercer & Miller, 1992; Miller, Harris, Strawser, Jones, & Mercer, 1998; Miller, Mercer, & Dillon, 1992; Peterson, Mercer, & O'Shea, 1988). Although the use of manipulatives and visual representation is sometimes viewed as a technique only used with younger children, studies demonstrate the value of continuing to incorporate concrete and visual representation with secondary students (Huntington, 1995; Maccini & Hughes, 2000; Maccini & Gagnon, 2000; Witzel, 2005).

Unfortunately, many textbooks designed for core instruction do not make adequate use of the CPA continuum to develop mathematical reasoning (Bryant et al., 2008; Hodges, Carly, & Collins, 2008; Gersten et al., 2009). The majority of textbooks reviewed do not consistently use concrete or pictorial representation when introducing new concepts. When they do include concrete and pictorial representations, they fade them too quickly (Alkhateeb, 2019). There is not enough repetition for struggling learners to consolidate their understanding. Therefore, interventionists frequently need to supplement commercial materials with additional examples using objects, pictures, graphs, and other visual representations.

Interventionists also need to highlight the relationship among the various forms of representation. Case (1998, p. 1) explains that "students with good number sense can move seamlessly between the real world of quantities and the mathematical world of numbers and numerical expressions." However, students who have difficulty understanding mathematical concepts often struggle to connect the various forms of mathematical representation (Hecht, Vogi, & Torgesen, 2007). While they may appear to understand a concept when it is represented concretely or using pictures or other visual representations, students who struggle with mathematics may fail to recognize how the representations relate to the same concept when presented in the form of abstract words or symbols. Because students' understanding of the relationship between representations has been linked to increased understanding of mathematical concepts, the IES Practice Guide identifies it as an evidence-based practice, stating, "We also recommend that interventionists explicitly link visual representations with the standard symbolic representations used in mathematics" (Gersten et al., 2009, p. 31). In other words, interventionists should introduce a concept concretely, then show students how that concrete representation connects to the visual and abstract representations, using consistent language to highlight the relationships. Figure 6.2 shows how a teacher could explicitly connect concrete, pictorial, and abstract representations during an introductory multiplication lesson. (The portion of the lesson shown in the figure illustrates the initial modeling provided during explicit instruction. The complete lesson would include all components of an explicit lesson.) Note that the teacher introduces all three forms of representation during this initial presentation. Based on Hudson and Miller's findings, the teacher would continue to include concrete representation in lessons for two more days. Then, if students demonstrated understanding, the teacher could discontinue the use of manipulatives and focus on pairing two-dimensional representations with the abstract labels for three more lessons. Finally, if students demonstrated understanding, the teacher could discontinue all visual representations and use only abstract numbers and words.

Scheuermann, Deshler, and Shumaker (2009) developed an instructional method that combines elements of explicit instruction with elements of inquiry-style instruction. They name their model the "Explicit Inquiry Routine." Instead of the teacher creating the initial

Figure 6.2 Explicitly Link CPA

CC.3.OA.1.Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. ***For example, describe a context in which a total number of objects can be expressed as 5 × 7.***	
Teacher says:	**Teacher Shows:**
I need your help to solve a problem. I'm going to have 3 children at my house tomorrow, and I want to give each of them 4 cookies. I need to figure out how many cookies I will need in all, so that I'll have enough to give each child 4 cookies.	
First I'm going to write my problem on the board. There will be 3 children, and each child gets 4 cookies. That is a multiplication problem. I will write it like this *(writes 3 x 4)* The first number, the 3, tells how many groups I have. I have 3 groups, one for each of the 3 children. The second number, the 4, tells me there are 4 objects in each group. Each child gets 4 cookies.	**3 x 4 =**
Concrete Representation: I have some plates here that I can use to show my problem. Each plate will represent one group. How many plates do I need? Yes, 3plates, one for each child. *(Puts out 3 plates)*	
Now I need to put cookies on each plate. How many cookies should I put on each plate? Yes, 4. *(Puts 4 cookies on each plate)* Did I show 3 x 4? Good.	
Pictorial Representation: I want to take notes to help me remember what I've done. I'm going to draw 3 circles on the board to represent the 3 plates. *(Draws 3 circles)*	
Now I'll put cookies on each plate. My drawing doesn't have to be fancy, so I'm just going to use lines to represent the cookies. *(Draws cookies)*	
Let's check. Did I show 3 x 4? Does my drawing match the plates of cookies on the table? Good!	**3 x 4 = 12**
Abstract Representation: Let's solve the problem. How can I determine how many cookies I'll need in all? Sure, I can count. Let's count together. *(Counts the 12 cookies with class)* I'll need 12 cookies. I'm going to write that in my equation like this *(Records the product in the equation)* 3 groups of 4 objects equals 12 objects. 3 x 4 = 12.	

model, the explicit inquiry routine gives students more input in the early stages of the lesson. The teacher asks a series of carefully scaffolded questions to lead the students through the representational process, monitoring their responses in an extended guided practice activity to ensure student understanding before moving into independent practice. First, the teacher poses a mathematical problem and engages the whole class in a discussion of ways to represent that problem. The authors label this initial portion of the lesson "tell me how." The class brainstorms ways to represent the problem using concrete objects. Once students have created a successful concrete model, the teacher asks them to brainstorm ways to represent the problem using visual representation. Then the teacher moves to abstract representation, asking the group to develop a number sentence that represents the problem. After the class successfully completes the "tell me how" segment, the lesson progresses to the "tell your neighbor how" phase. Each student turns to his or her neighbor and explains the graphic and symbolic representations the class developed. Finally, students have an opportunity to solve the problem on their own. They are instructed to talk themselves through

Figure 6.3 The Explicit Inquiry Routine

Kyle and Andrew have the same number of candies. Kyle has a bowl of candies and 4 additional candies. Andrew has 14 candies.

1. **Tell me how** . . .
   - How could we represent this situation using these objects?
   - How can we represent this situation using graphic pictures?
   - Is there another way we could represent this same situation?
   - How can we represent this situation using math symbols?

2. **Tell your neighbor how** . . .
   - Now tell your neighbor how you represented the situation concretely.
   - Tell your neighbor how to represent the situation using graphic pictures.
   - Can you explain the mathematical symbols you chose to use to your neighbor?

3. **Tell yourself how** . . .

   As you illustrate and manipulate this situation by yourself, get ready to explain your choices. You may want to talk yourself through the process because I may ask you why you chose to use what you did.

Reference: Scheuermann, A. M., Deshler, D. D., & Schumaker, J. B. (2009). The effects of the explicit inquiry routine on the performance of students with learning disabilities on one-variable equations. *Learning Disability Quarterly*, *32*(2), 103-120.

the process and to be prepared to explain their choices. Figure 6.3 shows the steps in the Explicit Inquiry Routine. In their research, when this procedure was used, student learning increased (Scheuermann, Deshler, and Shumaker, 2009). The carefully scaffolded questions help students develop their own representations and make connections among concrete, pictorial, and abstract models. Using the explicit inquiry routine allows students to be actively involved in figuring out ways to represent mathematical concepts, but still provides enough guidance that students who typically struggle with representation can succeed.

Whether using explicit instruction or the Explicit Inquiry Routine, when instructors clearly show students the relationship between different representations, achievement has increased. In many commercial programs, this is an area of weakness. When textbooks use concrete and pictorial representations, they seldom provide sufficient examples at each stage, and they seldom explicitly link the various forms. If teachers consistently use the CPA continuum to introduce mathematical concepts and procedures, and then explicitly link the different types of representation, they can increase learning outcomes. When the materials provided by the district do not explicitly link the concrete, visual, and abstract representations, then interventionists will need to add this component to their lessons.

Virtual Manipulatives

Modern technology now provides the option of using digital "objects." These virtual manipulatives are two-dimensional renditions of traditional manipulatives. Unlike other two-dimensional representations, however, students can manipulative the virtual objects to demonstrate mathematical concepts and procedures. Virtual manipulatives therefore fall between the "C" and the "P" in the CPA continuum (Moyer, Bolyard, and Spikell, 2002). The National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/vlibrary.html) provides a wealth of free virtual manipulatives appropriate for use in tiered interventions. NCTM's *Illuminations* (http://illuminations.nctm.org/) website also includes many virtual manipulatives, as does Slidsmania (https://slidesmania.com/tag/manipulatives/). Technology allows easy access to materials that may not be readily available in the classroom. When the application is displayed on a smartboard, the entire class can easily see the demonstration. Some students find these virtual activities especially motivating. In addition, virtual manipulatives can support online instruction. The online resources for Chapter 6 include a list of additional websites offering activities, lessons, and games using virtual manipulatives.

Research with struggling learners suggests that lessons that use concrete representation, whether with actual objects or virtual manipulatives, are more effective than lessons that do not incorporate concrete representation (Carbonneau, Marley, & Selig, 2013; Larbi & Mavis, 2016). In addition, research shows that incorporating virtual manipulatives can effectively augment instruction when used in combination with traditional manipulatives (Moyer-Peckenham, Salkind, & Bolyard, 2008; Reimer & Moyer, 2005; Steen, Brooks, & Lyon, 2006; Suh, Moyer, & Heo, 2005; Suh & Moyer, 2007). However, the research comparing virtual manipulatives to concrete manipulatives is inconclusive. Most studies show no significant difference between the two, but some found that students prefer concrete manipulatives (Burns & Hamm, 2011; Liggett, 2017; Satsangi, Bouck, Taber-Doughty, Bofferding, & Roberts, 2010). Older students, who sometimes view concrete manipulatives as babyish, may respond more favorably to virtual manipulatives. One study suggests that students on the autism spectrum may do better with virtual manipulatives (Bouck, Satsangi, Doughty, & Courtney, 2014). Current research has not been established that virtual manipulatives can replace three-dimensional objects to develop conceptual understanding. We recommend that students who require tiered interventions have opportunities for hands-on experiences with actual objects before they progress to the more abstract virtual representation, but also have an opportunity to work with virtual manipulatives. Whichever form is used, interventionists must remember to explicitly link the objects or virtual manipulatives with the standard symbolic representations used in mathematics.

Recommendations for Implementing the CPA Sequence

Here we summarize recommendations for incorporating manipulatives and pictorial representations into lessons. In subsequent chapters, we will provide more detailed discussion and examples of ways to use the CPA continuum when introducing specific mathematical content.

1. Let students use the manipulatives themselves. Simply having the teacher use manipulatives in a demonstration is insufficient. Students need hands-on practice.
2. When introducing new concepts and procedures, follow the CPA continuum. Begin with concrete representation, then progress to pictures, tallies, diagrams, and other two-dimensional representation, and then to abstract words and symbols. All three types of representation can be introduced in the same lesson, but students need several experiences with concrete representation before they are ready to discard the manipulatives and work solely at the pictorial level, and several more experiences before the pictorial representation can be faded and students are ready to rely only on abstract representation. Studies have shown that students with disabilities typically needed three lessons using manipulatives, and three more using pictorial representation, before relying solely on abstract words and symbols. Hudson and Miller (2006) suggest that students should be able to complete ten independent practice problems in order to progress from one level to the next. Interventionists can also assess student understanding by asking them to explain their representations. Students who can explain how various representations illustrate the same mathematical concept are ready to progress to the next level.

3. Explicitly link concrete, visual, and abstract representations, because students who have difficulty frequently fail to connect the various forms of mathematical representation. Explicitly linking the various representation systems, using consistent language across systems, and having students explain how the representations are connected, has resulted in higher achievement outcomes. Figure 6.1 provided an example of linking representational systems.
4. When selecting manipulatives, choose items carefully to clearly highlight the concept. It is not sufficient to simply give a student an object to move. The manipulative should provide a three-dimensional representation of the mathematical concept or procedure the students are learning.
5. Provide opportunities for students to model the same concept using a variety of different manipulatives and visual representations. The ability to represent mathematical ideas in multiple ways is a critical component of quantitative reasoning. For example, fraction circles are frequently used to model fractional parts of wholes, but students should not be limited to thinking of fractions as parts of circles. Their understanding will be enhanced if they also experience other examples, such as finding fractional parts of squares and rectangles, or using fraction bars, towers, and other manipulatives. Decimal values can be modeled with a variety of manipulatives, including using base-ten blocks, DigiBlocks, or metric weights. Graph paper and number lines allow students to create visual representations of decimal numbers. Using a variety of different manipulatives and visual representations to model the same concept deepens students' conceptual understanding.
6. Provide opportunities for students to translate among different representations. Students with rich number sense can fluently transition among all types of representations, but students who struggle to represent mathematical ideas may have difficulty making the same connections. For example, given the concrete representation of a mathematical expression, a student may be able to write a numerical expression to describe it. That same student may become confused when asked to reverse the process and, given the numerical expression, represent it with manipulatives. Sometimes, teachers routinely ask students to create one type of representation but neglect others. To develop a rich conceptual understanding, students need opportunities to practice converting among all representational forms. They should have opportunities to practice all of the following:
   - ♦ Given a concrete representation, model it using pictures, diagrams, and other visual representations, as well as with numbers and words.
   - ♦ Given a visual representation, model it using concrete materials, numbers, and words.
   - ♦ Given a numerical expression, represent it concretely and visually and explain it in words.
   - ♦ Given a word problem, represent it using concrete representation, visual representation, and with a numerical expression.

   These opportunities will develop students' ability to fluently transition among representations.
7. Provide opportunities for students to explain their thinking. For example, students could share with their classmates the process they followed to obtain their answer, or explain why they selected a particular strategy, or they could explain their thoughts in a math journal or use a diagram to explain how they approached a particular problem. Asking students to explain their work helps consolidate understanding and also allows

interventionists to assess students' understanding. The National Council of Teachers of Mathematics (NCTM) recommends that all students be provided opportunities to:

- ♦ Organize and consolidate their mathematical thinking through communication
- ♦ Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- ♦ Analyze and evaluate the mathematical thinking and strategies of others
- ♦ Use the language of mathematics to express mathematical ideas precisely. (NCTM, 2000, p. 128)

8. Use manipulatives judiciously and systematically fade their use. Manipulatives provide an excellent foundation for understanding mathematics, but the goal is that students will become proficient with standard symbolic representation and not remain dependent on concrete supports. If students continue to work with concrete objects, they may not develop the ability to function at the abstract level. Interventionists should systematically fade the use of manipulatives and help students transition to visual and abstract representation. Research with students with disabilities suggests that three experiences with manipulatives is usually sufficient to develop initial understanding. As soon as students are able, fade manipulatives and focus on pictorial and abstract representation.

Following these suggestions when incorporating the CPA continuum into lessons can increase success in tiered interventions.

Summary

The ability to represent mathematical quantities in multiple ways is a critical component of quantitative reasoning. Representation allows students to organize mathematical information, describe mathematical relationships, and communicate mathematical ideas to others. Conceptual understanding of quantity follows a developmental sequence, beginning at the concrete level with physical actions and three-dimensional objects. As their understanding deepens, students progress to using pictorial representations such as charts and diagrams to model mathematical relationships. If these concrete and visual representations are linked to more abstract words and symbols, students eventually can use words and symbols meaningfully without needing the concrete and pictorial representations. The CPA continuum is an evidence-based practice recommended for students receiving tiered interventions.

Since textbooks used in the core curriculum usually provide limited concrete and pictorial examples, and seldom show students how the various representations are related, interventionists will frequently need to intensify instruction by adding these components to the lesson. In the next chapters, we provide more detailed descriptions of ways to incorporate concrete and pictorial representation to help students master whole numbers and rational numbers.

7

Developing Number Sense

The National Council of Teachers of Mathematics identifies developing students' understanding of numbers and their relationships, representing numbers, and understanding number systems, as the focus for pre-kindergarten through second grade mathematics instruction (NCTM, 2000). Numerous studies have found that early number sense predicts later mathematics achievement (see for example, Duncan et al., 2007; Jordan, Glutting, & Ramineni, 2010; Jordan & Levine, 2009; Tostok, Petrill, Malykh et al., 2017; Woods, Ketterlin-Geller, & Basaraba, 2018). In this chapter, we will focus on counting and representing whole numbers using objects, visual representation, words and symbols, as well as comparing magnitude, and understanding the place value of numbers in the base-ten system. In Chapter 8, we will begin to look at operations with whole numbers.

Counting and Representing Whole Numbers

The ability to count meaningfully and to understand relationships among numbers is a critical early childhood skill. A child whose number sense is well-developed can count a set of six objects and state the total. The child can also represent the number six or recognize the quantity when it is illustrated using a variety of dot patterns or objects, knows that 6 is more than 5 and less than 7 and that it is composed of 3 and 3 and also of 2 and 4, and can identify real-world applications of six. While many students enter school with highly developed number sense, children who experience difficulty with mathematics may need intensive support to attain the same level of understanding.

In order to count meaningfully, students must master two separate skills: *rote counting* and *one-to-one correspondence*. Rote counting is the ability to state the number words in order (i.e., one, two, three, four, …). Young children can often recite the counting sequence from memory but cannot count meaningfully because they do not realize that each number in the sequence represents one and only one object. When they can point to one object and say one number, and then wait to say the next number until their fingers touch the next object, we say they have one-to-one correspondence. Students who lack one-to-one correspondence will not only be unable to count a group of objects or progress in mathematics, but they will also have difficulty reading, because letter-sound correspondence is based on the idea that each letter or group of letters represents one sound or word. In other words, one-to-one correspondence is a pre-requisite for academic progress in both mathematics and reading.

Concrete experiences are essential for students to develop one-to-one correspondence (Frye, Baroody et al., 2013). Activities that involve matching actual objects, such as giving one cookie to every child in the group or putting one hat on each stuffed animal, are necessary in order for students to learn to pair each object with a single number name, and each number name with a single object. Counting activities that involve large muscle movement are especially helpful, because the motor activity helps separate each item and so highlights the discrete value of the objects. For example, if the child must carry one item from the desk to a classmate and say "one," then come back and get a second item and walk to the next classmate before saying "two," the relationship between the spoken number and the physical object becomes more obvious. Students need multiple experiences like this where they move and use large muscles to match concrete objects before they will make the cognitive connection that one word is associated with just one object. Once they can count successfully when engaged in large motor activities, then they can begin to count objects that are placed more closely together, such as small objects placed on a desktop, and then transition to counting pictures, and eventually students can work with only with abstract numerals.

By the end of kindergarten, typically developing students are expected to count and recognize numerals to 20; by the end of second grade they count and read numbers to 120. Second grade students work with numbers to 1,000, and by fourth grade students are expected to work with numbers to 1,000,000 (National Governors Association, 2010). Research studies demonstrate that students who struggle academically need more systematically designed instruction that their typically achieving peers in order to meet grade-level expectations (McLeskey et al., 2017). They benefit when educators allow them to master pre-requisite skills before introducing higher level skills, and when similar skills are introduced separately before students are expected to discriminate among them (McLeskey et al., 2017). For example, a student who struggles with number sense should have already mastered the numerals 1-5 before the teacher introduces the numeral "6," and then the student should master "6" before another numeral is introduced. Core materials often introduce more than one numeral in a lesson, and then add another numeral in the next lesson. In contrast, an interventionist should monitor student performance and use that information to gauge when to introduce new content.

A variety of manipulatives are available to help students develop a robust understanding of numbers and their values. We describe several here.

Counters

The concrete objects that students manipulate as they develop beginning number sense are often referred to as "counters." Any toy or small object that appeals to students can be used to help develop quantitative reasoning. Skittles, M&M's, Hershey's Kisses, gummy bears, and other types of candy can instantly capture children's attention while effectively representing mathematical concepts and procedures. Small toys such as miniature cars, bears, or dinosaurs intrigue children. Natural objects such as pebbles and shells or familiar household objects such as buttons, coins, bottle caps, poker chips, or paper clips can be equally effective. Unifix cubes or pop cubes are commercially available counters that can be snapped together, so they have the added advantage of allowing students to connect cubes when modeling larger numbers.

Counters are commonly used in elementary mathematics classrooms, and commercial programs provide excellent ideas for using counters to develop quantitative reasoning. Since most educators are familiar with these manipulatives, we will not describe their use in detail here. The big ideas for interventionists include: (1) introduce numbers systematically (i.e. pace the introduction of new numbers to allow students to master lower numbers before moving on), and (2) follow the CPA sequence, explicitly linking the concrete counters with visual and symbolic representations. Core textbooks frequently have students model a concept or procedure with counters and then immediately move to performing the skill using pictures and then numbers, but without showing students how the various representations are connected. As we discussed in Chapter 6, although normally achieving students may be able to successfully transition from one form of representation to another without the need for scaffolded support, students who have difficulty with number sense often struggle to connect the various forms of mathematical representation (Hecht, Vogi, & Torgesen, 2007) and so benefit when these connections are made explicit. For example, if students initially used M&M's to count, they could use a pencil, crayon, or chalk to draw a model of their M&M's. Recording the written numeral on their drawing helps students connect the three-dimensional objects with pictorial and symbolic representations. Written numerals are at the abstract end of the CRA continuum, so students will need multiple experiences pairing these abstract representations with their concrete and pictorial equivalents before they can work meaningfully with just the abstract numerals. Students must be able to move back and forth among representations, going from concrete to representational to abstract and also from abstract to representational to concrete. Given a group of objects, students should be able to draw a picture to represent the group, represent it with tally marks, say the number name associated with that quantity, and write the numeral. Given the pictorial representation, they need to practice illustrating it with objects, tally marks, words, and numerals. When told the name of a numeral, they should be able to write it and model it concretely and pictorially. Given the numeral, we need to ask them to read it aloud and model it with objects and drawings. The ability to represent mathematical quantities in multiple ways is a critical component of quantitative reasoning, but individuals who struggle with mathematics often have extreme difficulty connecting the various types of representations. To help struggling learners develop a robust number sense, interventionists often need to add multiple opportunities for students to represent numbers in a variety of ways, and to talk about the similarities and differences among the representations.

TouchMath

TouchMath (www.touchmath.com) is a system specifically designed to help students associate written numerals with the quantities they represent. Students first learn to place counters in a set pattern on large drawings of numerals. In other words, they learn to place one counter at the top of the written numeral 1, two counters at specific locations on the numeral 2, and so on. Eventually they transition to using dots drawn on the numerals instead of actual counters, and finally the dots are faded and students mentally count the places where the dots would have been placed in order to identify the value of the numeral. The system provides very effective support that explicitly links concrete representation, visual representation, and abstract numbers. However, students sometimes become overly dependent on counting dot patterns and do not make the transition to purely abstract representation. Research with students with learning disabilities found that most students need about three lessons that include concrete representation, followed by three more lessons using pictorial or graphic representation, before these supports can be faded and students can work meaningfully at the abstract level (Hudson & Miller, 2006). Students who have learned to use TouchMath often continue to count dots long past the seventh lesson. We recommend that TouchMath be used judiciously because students who continue to rely on counting the dot patterns are likely to have difficulty later transitioning to more efficient abstract representation.

Ten Frames

One of the best tools to help students connect three-dimensional concrete representation to two-dimensional visual representation is the ten-frame. A ten-frame consists of an empty 2x5 grid onto which students place counters. See Figure 7.1. The structure, generally attributed to Robert Wirtz (1974) and further developed by Van de Walle (1988) and Bobis (1988), is now included in many of the programs adopted for use in core instruction. A smaller 1x5 grid, or five-frame, is often introduced initially to help students master numbers to 5, followed by the larger 2x5 grid when students progress to numbers from 6 to 10. The frames can be purchased commercially or made from tagboard. In addition to the boards illustrated in Figure 7.1, manufacturers offer ten-frame trains, which are three-dimensional

Figure 7.1 Ten Frames

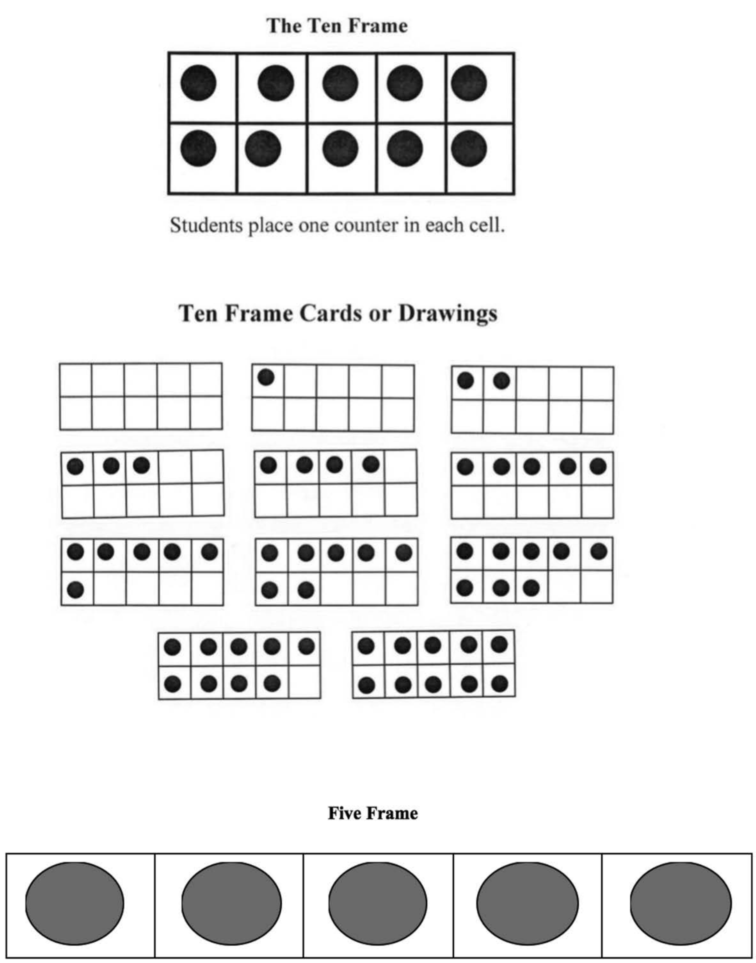


Figure 7.2 Ten-Frame Train

versions of the ten-frame grid designed to look like train cars. Students place teddy bear counters or other objects into the train's ten compartments to model quantities to ten, or connect multiple train cars to model larger numbers. Teachers can make their own versions of a ten-frame train by cutting the end off of an egg carton, leaving two rows of five egg cups that form the train's ten compartments. Figure 7.2 shows a ten-frame train.

When they are first learning to model numbers with frames, students who are struggling should be encouraged to lay the frame horizontally so that there are five boxes in the top row. They begin by placing the first counter in the upper left corner and progress from left to right across the top row, then move to the bottom row and continue placing counters from left to right, just as the eyes move when reading. Placing counters on the ten-frame in this set order helps students organize their counting and develop a mental model of each quantity. Once students master recognizing numbers when the counters are arranged in the left-to-right order described above, they should also work with ten-frames where the counters are arranged in random order to solidify their understanding that rearranging the counters does not affect the total quantity.

In addition to using the ten-frame to model a given quantity, students can practice adding one and then two more counters to the board and stating the new number. Students whose sense of number is still developing will start from one and recount, but as they become more proficient they learn to "count on" from the last number stated. "Counting on" is a skill that students will need when they begin addition. Students can also remove counters while counting backward in order to prepare for subtraction.

The spatial organization of the ten-frame supports students' emerging number sense because the frames provide a graphic illustration of a number's relative value. Representing 8 with 5 counters on top and 3 below clearly shows that 8 is "5 and 3 more" and also that it is 2 less than 10. When students place a given number of counters on the frame, and then count to determine how many more counters would be needed to fill the frame, they begin to recognize combinations that make ten. After students master counting by ones, the frames can be used to model skip counting. Using multiple five-frames can help students learn to count by fives, while using multiple ten-frames provides a model for counting by tens. The frames can also be used to introduce coin values. Students can place pennies on a five-frame until all the squares are filled, and then exchange their five pennies for a nickel. In a similar manner, placing pennies on a ten-frame provides a model for the value of a dime.

Because the concrete and pictorial versions of the frames are so similar, ten-frames facilitate the transition from concrete to visual representation. At the concrete level, students directly manipulate objects when they place counters on the ten-frame board. At the visual representation level, students can draw the counters on a blank board or use pictures of ten-frame boards containing pre-drawn dots, like those shown in Figure 7.1. These pictures can also be made into playing cards to practice initial counting skills and comparing numbers. The graphic organization of the ten-frame highlights the benchmark numbers of five and ten and therefore helps build number sense in learners struggling with initial mathematical concepts. See the online resources for websites that offer blackline masters of ten-frame boards and numerous games and activities using ten-frames at both the concrete and pictorial levels.

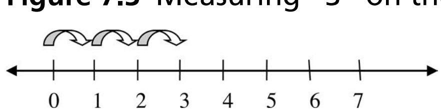
Number Lines

In addition to using objects and pictures to model numbers, students must learn to use a number line diagram. Number lines show the relative values of all real numbers, including whole numbers like two and 327, which are the focus in this chapter, fractions and decimals such as ½ or 14.268, which we will discuss in Chapter 10, and negative numbers such as –4 or –13/8. Number lines are frequently used with young students to develop initial counting and later to model more advanced concepts. The Common Core State Standards specify that students should be able to use number lines by the end of the second grade:

> CC.2.MD.6: Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, …, and represent whole-number sums and differences within 100 on a number line diagram. (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010)

Students who struggle with mathematics sometimes have difficulty using number lines effectively, because instead of counting *objects* as they have done previously, when students use a number line they must begin at zero and count the number of *spaces* as they move forward. See Figure 7.3. Many students find counting spaces cognitively challenging. They may focus on the numbers or points on the line, not the spaces between. Focusing on the points can sometimes lead to difficulty when students later attempt to use number lines to solve addition and subtraction problems. For example, to model the problem 8 – 2, a student should begin at 8 and move one space to the left, touching 7. That movement represents the first space. He then jumps back to 6, which is the second space. Students who follow this process obtain a correct answer of 6. Students who focus on counting the points might approach the same problem by placing a finger on the 8 while counting "One," and then move to the 7 and say "Two. 8 – 2 = 7." The student might make a similar error when using a number line to model an addition problem such as 2 + 3 by beginning at the point representing the first addend, 2, and counting up from there. This incorrect process would result in an answer of 4, which is one less than the actual sum of 2 + 3.

Figure 7.3 Measuring "3" on the Number Line



Some of these difficulties can be prevented if the students' experience with number lines follows the CPA sequence. The number line is a type of diagram with abstract numbers arranged to show their relative values. In other words, a number line addresses the representational and abstract levels of the CPA continuum. In order to experience this structure at the concrete level, students can walk along a large number line taped on the floor, counting their steps as they move. The large muscle motion helps them focus on counting spaces. They can then progress to using a smaller number line taped on the desk, and count spaces by demonstrating a bunny hopping or a frog jumping down the line. These concrete experiences can help ensure that students will be able to use the number line effectively in the future.

MathLine (www.howbrite.com/) is a manipulative that adds concrete representation to the visual and abstract representation inherent in every number line. It consists of a three-dimensional number line with movable plastic rings that can be used to illustrate whole-number values. All the rings are contained within the MathLine frame, so there are no loose pieces for students to misplace, and manipulating these self-contained rings helps organize the counting process. See Figure 7.4. The rings are white, except for multiples of 5, which are blue, and multiples of 10, which are red. To illustrate 5, the student would begin with all the rings pushed to the right end of the MathLine, and then slide five rings all the way to the left. Each ring fills one space on the number line, so when the rings are pushed up against the zero mark on the left side of the device, the total value of 5 is seen just to the right of the fifth ring, as shown in Figure 7.4. To add on two more, the student would simply slide two additional rings to the left, resulting in the sum of 7 showing to the right of the seventh ring. This concrete action combines the tactile experience of touching the rings and the kinesthetic movement of sliding the rings with the visual representation of abstract numerals arranged in order on a number line. MathLine therefore combines the complete CPA continuum in a multimodality experience that can help build conceptual understanding. Because it highlights the benchmark numbers of 5 and 10, MathLine also provides some of the advantages of a ten-frame. For example, since the fifth ring is colored blue, students

Figure 7.4 Math Line

can easily see that the quantity 7 is "5 and 2 more." The tenth ring is red, so students can also see that 7 is "3 less than 10." In addition, the colored rings facilitate skip counting by 5 and 10 and are also helpful when teaching students to round to the nearest ten. The company's website offers video clips that illustrate using MathLine to enrich students' understanding of counting and rounding whole numbers and decimals, as well as performing operations with whole numbers.

When students have the opportunity to represent their thinking using a variety of different manipulatives, their understanding deepens. Allowing students to work with both counters and number lines will help them develop a richer understanding of mathematical concepts and procedures.

Magnitude Comparison

In addition to reading, writing, and representing numbers, students with robust number sense have a strong understanding of magnitude comparison. They can look at two sets of objects and identify which set has more or fewer objects, and they can compare two numbers and identify which represents a larger or smaller quantity. The ability to compare values progresses developmentally. Piaget (1965) noted that if young children are shown two parallel lines of counters that are the same length, they will recognize that they are equivalent. However, if the counters in one line are spread out or compressed, the child will say that there are more counters in the longer line, or fewer counters in the shorter line. According to Piaget, children begin to conserve number around age six. Students who can conserve number will compare the sets by counting and matching the objects in each line to determine which has more, or explain that the quantity must be the same, even though the objects were rearranged, because nothing was added and nothing was taken away. Providing opportunities for children to explain their reasoning helps consolidate learning, while also allowing the interventionist to assess the student's understanding.

Children need many experiences comparing objects in order to develop a robust sense of number magnitude. Following the principles of systematic instruction, it is best to initially present objects in two straight lines. Comparing sets of objects that are arranged randomly is a more challenging task, and should be delayed until students can accurately compare two lines of objects. Unifix cubes, which can be snapped together, are ideal for initial comparisons, because placing two lines next to each other creates a clear model of the relative values of the two sets. Although we are not aware of definitive research on this, anecdotal reports suggest that students grasp the concept more quickly when the lines are arranged vertically to show a taller and a shorter line, rather than being arranged horizontally, which requires students to compare widths.

Academic language is important. The language used for initial magnitude comparison begins with simple words like more/less, bigger/smaller, or taller/shorter. To intensify instruction for students who require tiered interventions, experts recommend that interventionists emphasize and repeat academic language (Powell & Fuchs, 2015). In addition, emerging research suggests that adding gestures can help students understand and remember verbal language. Gestures support working memory and help students make connections, which improves academic achievement (Goldin-Meadow & Alibali, 2013; Hord, Marita, Walsh, Tomaro, Gordon, & Saldanha, 2016; Walsh & Hord, 2019). To illustrate the difference between two sets, place your hands on top of each other in front of your chest, with one palm resting on the back of the other hand, and then raise the top hand up and down as you describe the sets. The top hand represents the larger quantity, while the bottom hand represents the smaller quantity. The space created as your hands move up and down represents the difference. Frequent repetition of both the words and the gestures may help struggling learners master magnitude comparison.

To help students connect comparing objects to comparing abstract numerals, students can place two MathLines (described above) next to each other to visualize relative magnitude. They can also arrange Unifix cubes in a number line frame. The frame holds the cubes in place, and has numerals printed on it spaced appropriately so students can see the numeral that matches the length of their cube train. Double number line frames are available that contain two rows of number lines arranged parallel to each other, so students can arrange a set of cubes in one row to illustrate one number, such as seven, and then arrange another set of cubes in the adjoining row to represent a different value, such as five. The frame holds the cubes in place and allows students to compare the two values. A game version of the double number line, called Mini Motor Math (available from www.learningresources.com), uses a racetrack theme and brightly colored miniature cars to compare quantities, which may help motivate reluctant learners.

Students in kindergarten initially use the words "more/less" or "bigger/smaller" to compare up to ten objects or pictures, using matching and counting to determine relative values. Later, they learn to use comparison language (i.e., the words "is less than," "is greater than," or "is equal to") to compare up to ten objects or pictures, and still later they compare values between one and ten presented as written numerals. In first grade, students are introduced to comparison notation (i.e., the symbols >, <, =), and they learn to write the math expression and read it using standard comparison language. As students gain proficiency with larger numbers, they use place value to compare numbers.

Place Value

In order to work meaningfully with quantities larger than ten, students need to develop an understanding of place value in the base-ten system. Where a digit is placed within a number tells its place value. When we write the numeral 12, the digit one represents one group of ten, and the two indicates two additional units. Nothing in the number word "twelve" suggests to a young child that it is equivalent to 10 plus 2, so American teachers must devote a great deal of instructional time to helping students understand place value. Most programs designed for use in the core curriculum include base-ten blocks or other manipulatives, and most programs contain excellent ideas for using them to model place value and expanded notation. In their eagerness to help students master more advanced concepts, teachers sometimes skip over these activities. However, research has demonstrated the value of ensuring that students use concrete and pictorial representations to model their understanding of numbers and the base-ten system. Spending time modeling expanded notation with base-ten blocks and other manipulatives will actually save time later, because students with a robust sense of place value will experience fewer difficulties when they encounter more advanced operations with larger numbers. For example, consider the following addition error, which is commonly made by students who struggle with mathematics:

$$\begin{array}{r} \mathbf{74} \\ \mathbf{+56} \\ \hline \mathbf{1210} \end{array} \qquad \begin{array}{r} \mathbf{67} \\ \mathbf{+18} \\ \hline \mathbf{715} \end{array}$$

A student who records all the digits without regrouping does not understand the important role of place value in the base-ten number system. Errors such as these can be prevented if students have sufficient opportunities to experience concrete and visual representation of two- and three-digit numbers, to use place-value mats, and to express numbers in expanded form before they tackle advanced operations. When mathematical words and symbols are firmly rooted in experiences with concrete and visual representation, students find them meaningful. When students lack a foundation in concrete and visual representation, their attempts to perform symbolic operations may become a rote execution of meaningless procedures. A variety of manipulatives are available to help students understand place value.

Base-Ten Blocks

Base-ten blocks can be used to model ones, tens, hundreds, and thousands. The blocks consist of individual units (1x1 cm. cubes), "rods," or "longs" composed of ten unit cubes, "flats" formed from 100-unit cubes arranged in a 10x10 array, and a large cube containing ten flats which represents 1000. See Figure 7.5. The pieces are proportional, which means that a rod is ten times larger than a unit, and a flat is ten times larger than a rod. Base-ten blocks therefore provide an excellent model of the relative values of different numbers. Students can use base-ten blocks to model any number. Figure 7.6 shows how base-ten blocks would be used to model the number 345. Using blocks to demonstrate the value of two- and three-digit numbers, and then writing the numbers in expanded notation format (300 + 40 + 5),

Figure 7.5 Base-Ten Blocks

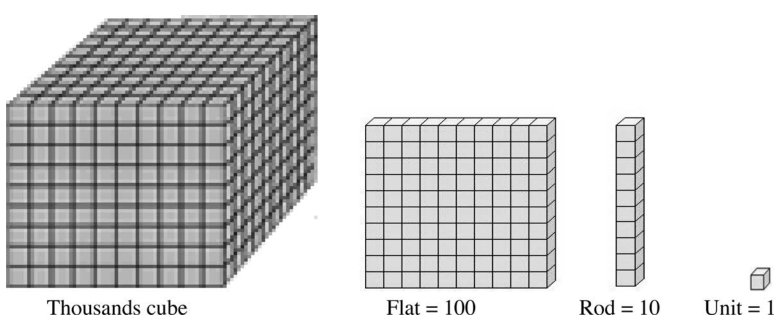


Figure 7.6 Representing 345 with Base-Ten Blocks

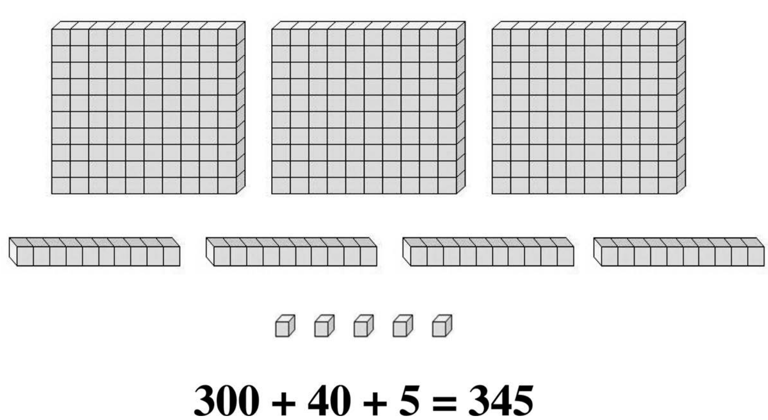
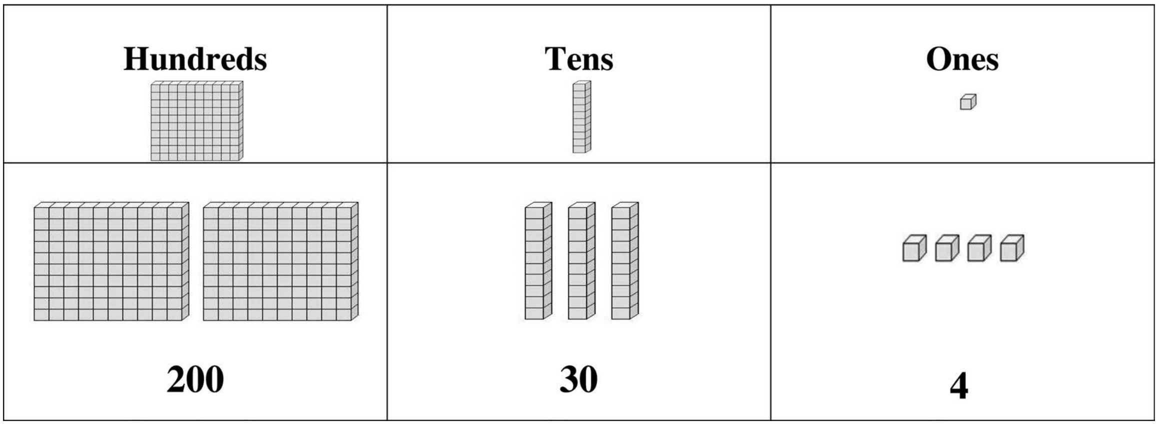


Figure 7.7 Representing 234 on a Place-Value Mat



helps students develop a more complete understanding of the quantities involved in two- and three- digit numbers. To further solidify their understanding, students can arrange the blocks on a place-value mat, which is a paper or plastic mat divided into columns, with each column labeled with a different place value. Students working with numbers to 99 can use a two-column mat that is labeled "ones" and "tens," and more columns can be added when students progress to larger numbers. Figure 7.7 shows a place-value mat with blocks representing the number 234.

When students use abstract numbers to express larger quantities, the column position indicates the place value. Place-value mats help students transition from concrete blocks to abstract numbers because the mats provide a two-dimensional graphic representation that forces learners to organize the blocks in the same order they will use when writing numbers. Initially, students place actual blocks on the mats. When they are ready to progress to the visual representation level, they can simply draw the blocks on the mat. Explicitly linking the blocks on the mat with the abstract symbols in both expanded notation and standard formats will help students understand place value within the base-ten number system.

To solidify their understanding of place value, students can play the Making Trades game which is available online. Players take turns rolling a die, collecting the designated number of unit cubes, and placing their cubes on a place-value mat. When they have accumulated ten or more units, they "make a trade" and exchange ten units for a rod, which they place on the mat in the tens column. The first player to accumulate ten rods exchanges them for a flat, places the flat in the hundreds column of the place-value mat, and wins the game. Playing the Making Trades game helps students understand expanded notation and the critical role of place value; it also lays the groundwork for future lessons involving the standard algorithm for regrouping in addition. The game can also be played in reverse: students begin with a flat and take away units and rods. In this version, the first player to run out of blocks is the winner. When the game is played in reverse, students are practicing the type of regrouping that will be required later in order to subtract multi-digit numbers.

Base-ten blocks made from wood, plastic, and compressed foam are available from most teacher supply stores. Instructors can also make their own base-ten blocks from mount board, poster board, or foam sheets. Flats, rods, and units can be cut using templates or a die-cut machine. In addition to the many books and programs that use base-ten blocks to help students understand place value, a multitude of free resources are available on the Internet. They include templates for making base-ten blocks, ideas for using blocks to develop initial number sense, virtual base-ten blocks that teachers can use to create lessons, applets for games featuring virtual base-ten blocks, and also ways to use base-ten blocks when performing operations with whole numbers, which we will address in Chapter 8. The National Center on Intensive Intervention provides an excellent template and video tutorial for using virtual base ten blocks during interventions (https://intensiveintervention.org/resource/virtual-lesson-example-show-me-number-using-base-ten-blocks). A variety of additional internet resources for base-ten blocks are available in the online materials.

KP Ten-Frame Tiles

A variation of base-ten blocks called KP Ten-Frame Tiles (https://kpmathematics.com/) can also help students understand place value. The student snaps individual unit tiles into a ten-frame that resembles a plastic version of the traditional ten-frame card. Because the individual tiles snap into a clear plastic cover, it is easy for students to see the relationship between individual tiles and a group of ten. In addition, ten groups of ten can be snapped into a larger see-through square that resembles a "flat" in traditional base-ten blocks. The website includes excellent brief videos that demonstrate how to use the tiles to model counting, place value, and operations.

DigiBlocks

Another manipulative that can help students understand place value is called the "DigiBlock" (www.digi-block.com/). DigiBlocks use small rectangular blocks to represent individual units, or "ones." These small blocks can be packed into a holder designed to contain ten individual blocks. When students put a lid onto the holder, they create a ten-block that is analogous to the rod used in base-ten blocks. The block of ten looks just like the single DigiBlock, except that it is ten times the size of the smaller block. See Figure 7.8. The ten-block holders can be grouped together into an even larger holder to form a bundle of 100, which is comparable to the flat used in base-ten blocks. The company also offers an enormous holder designed to contain ten blocks of one hundred, or 1,000 individual blocks.

Figure 7.8 DigiBlocks

This large holder is about one and a half feet tall and weighs about eighteen pounds, so it provides a powerful model of the relative size of 1,000. The holders that contain individual blocks can be secured onto a place-value board, called a "counter" by the manufacturer, and then individual blocks can be inserted into the holders to model place value. The second picture in Figure 7.8 shows a student using a DigiBlock counter to model tens and ones. Larger counters are available that can model larger numbers.

What sets DigiBlocks apart from other base-ten manipulatives is their ability to dynamically model the regrouping process. Students can insert up to nine blocks into the holder when it is secured in the ones column, but adding a tenth block releases a spring and causes the entire container to slide down the ramp. This provides a dramatic reminder that a maximum of nine units can be placed in the ones column. When the tenth block is added, the holder containing a complete group of ten must be transferred from the ones column to the tens column. A similar process occurs when students attempt to place a tenth bundle in the tens column; the holder with its complete group of one hundred must be moved to the hundreds column. To help students link the concrete blocks with abstract symbols, the holder includes a whiteboard where students can write the abstract number that represents the blocks in each column. Flip cards are also provided so that, instead of writing the numbers, students can simply display the appropriate value, as the student has done in Figure 7.8. The flip cards contain only the single digits 0-9, so they cue students that a "trade" is needed before recording larger numbers. DigiBlocks also offer miniature blocks that represent decimal tenths. Ten of the small blocks are equal in size to one single unit block, providing a clear model of the relative size of decimal tenths compared to the value of a whole number. Students generally enjoy playing with DigiBlocks, and their active engagement facilitates learning.

Place-Value Disks

Place-value disks are a popular new tool used to model place value in some basal math programs. They consist of round disks in a variety of colors. Each color represents a different place value, and the disks are stamped with the numerals 1, 10, 100, and 1,000 to indicate their value. Place-value disks are concrete objects that can be manipulated, but unlike base-ten blocks, they do not provide a clear model of relative values, because all of the disks are the same size. Therefore, they should be introduced only after a student has developed a robust understanding of place value through modeling with base ten blocks, KP Ten Frames, Digi-blocks, or other manipulatives that physically illustrate relative values.

Household Objects

In addition to ten-frames, base-ten blocks, and DigiBlocks, numerous other objects can be used to model place value. Popsicle sticks, coffee stirrers, or straws can be bundled together to show a group of ten. When students physically count ten sticks and bundle them into a group of ten, the relationship between a single unit and a group of ten becomes meaningful. Similarly, gathering ten groups of ten into a bundle of one hundred clearly illustrates the relative values of ones, tens, and hundreds. Pop cubes, plastic links, or paper clips can be connected to form a chain of ten or one hundred. Any of the counters students have used to develop initial number sense can be placed in small containers to create groups of ten and larger containers to form groups of one hundred. Providing students with opportunities to represent the same concepts with a variety of different materials will help solidify their understanding of place value.

Intensifying Instruction

The methods recommended for use in interventions differ from the methods used in the core curriculum in three important ways. First, while the core curriculum should balance the use of student-directed and teacher-directed instruction, students who struggle with mathematics benefit from teacher-directed presentation of carefully sequenced concepts and skills (McLeskey et al., 2017). Second, while general education teachers are encouraged to use rich, complex problems to engage students, studies show that struggling learners benefit from systematic instruction, where teachers introduce content in small, carefully sequenced chunks (McLeskey et al., 2017). Third, while mathematically proficient students may master content despite limited experience with concrete and visual representation, students who struggle learn more when the CPA continuum is carefully followed and the connections between representations are made explicit (Gersten et al., 2009). Studies show that the CPA sequence is not adequately addressed in most commercial materials, and even those materials that incorporate multiple concrete and pictorial examples seldom *explicitly* link the representational systems.

A number of validated programs have been developed to support students who require Tier 2 interventions (see Chapter 12 for sources for high-quality programs). When a validated program is implemented with fidelity, students who require Tier 2 supports should make progress. If they do not make adequate progress, then the interventionist may need to intensify instruction to meet the student's individual needs. Many interventionists do not have access to a program that is validated for use with students who require tiered supports, however. Teachers are often given materials designed for core instruction and expected to use them with students who require interventions. In addition, when students who need support are included in general education math classes, educators may prefer to adapt the core materials so that math support aligns closely with the materials, terminology and approaches being used in the general education instruction. When the materials used for Tier 2 have not been validated as effective with students who need interventions in mathematics, interventionists will need to intensify instruction to meet the students' instructional needs. In other words, they must adapt existing programs by adding evidence-based practices in order to more effectively address a student's targeted needs. The following list summarizes several evidence-based ways to intensify instruction for students who are struggling to develop robust number sense.

1. Make instruction more systematic. Break objectives into smaller segments, and sequence them carefully. For example, a basal textbook may introduce several numbers in a single lesson, while the student who requires tiered supports may need to focus on one new number at a time. The textbook may introduce all three symbols used in comparison notation (>, <, =) at the same time, while an interventionist may need to introduce each symbol in a separate lesson. The textbook may introduce several columns in a place-value chart at one time, while the student who struggles with number sense may need extended instruction with smaller numbers before another digit is added to the place-value chart.
2. Make the lesson more explicit. Most math textbooks do not use explicit instruction, so the interventionist may need to add teacher models, additional examples, and guided practice activities. Each student should have many opportunities to respond and receive immediate positive and corrective feedback before being asked to complete independent practice activities. Follow the suggestions in Chapter 5 to make the lesson more explicit.

3. Follow the CPA sequence, and explicitly link the various forms of representation, as discussed in this chapter and in Chapter 6.
4. Slow the pace of instruction. Students who struggle with mathematics often need many more practice activities to master a concept than are provided in a typical textbook. Even when students appear to understand a skill or concept at the end of a lesson, that does not mean that the new knowledge has moved from working memory into long term storage in the brain. Providing additional practice opportunities, and frequent review, promotes long-term retention.
5. Monitor progress and adjust instruction accordingly. If a student has not mastered a concept, do not simply move on to the next lesson in the text. Instead, re-teach the content until the student is successful.

In this chapter, we focused on using systematic, explicit instruction and the CPA continuum to help students develop number sense for whole numbers. In Chapter 8, we will show how the same principles apply to performing operations with whole numbers.

8

Operations with Whole Numbers: Addition and Subtraction

The term "operation" refers to a mathematical process used to calculate value. There are four basic operations: addition, subtraction, multiplication, and division. Because a student's ability to perform mathematical operations depends on a robust number sense, the activities described in the previous chapter are foundational for mastering operations. Students must understand the value of numbers before they can add or subtract meaningfully, and they also need to understand place value in order to perform operations with larger numbers. In kindergarten, most curricula focus on developing basic number sense, and then introduce the concepts of addition and subtraction within ten using objects, fingers, drawings, sounds, movement, and eventually equations. By the end of first grade, students are expected to add and subtract within 20, and are beginning to perform operations within 100. By the end of second grade, they should master addition and subtraction concepts, and by the end of fourth grade, they should be able to solve addition and subtraction problems fluently (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). Number sense and operations with whole numbers are the foundation for all higher mathematics, and so the IES Practice Guide recommends that "instructional materials for students receiving interventions should focus intensely on in-depth treatment of whole numbers in kindergarten through grade 5" (Gersten et al., 2009, p. 6). In this chapter, we focus on addition and subtraction, specifically developing conceptual understanding of addition and subtraction and modeling procedures for solving multi-digit addition and subtraction problems. In Chapter 9, we will discuss multiplication and division. Chapter 10 contains an in-depth discussion of methods for developing computational fluency with basic facts.

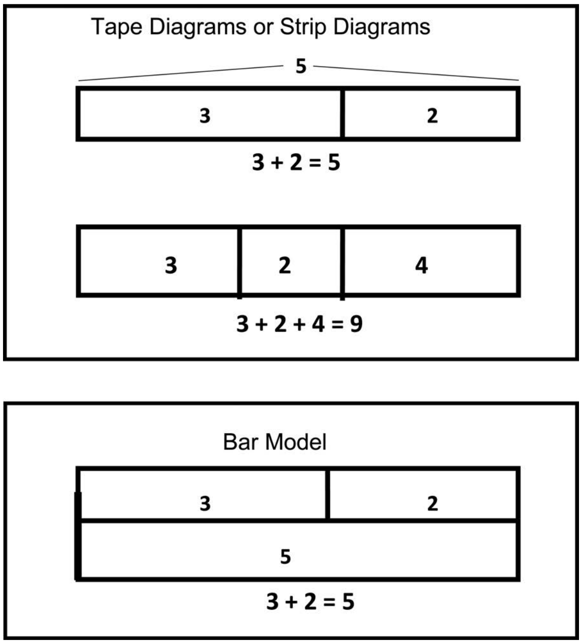
Developing Conceptual Understanding of Addition and Subtraction

Addition and subtraction problems are a natural extension of the counting activities used to develop initial number sense. For example, when students are learning to count, they often model a number by combining pop cubes or Unifix cubes. If their "train" contains cubes of different colors, they have created an addition problem. Although students who are still learning to count are not ready to use written notation like a plus sign or an equal sign, engaging them in discussions about the many ways to model a given number builds a foundation for more formal work with addition later. We might ask students to find different ways to combine two different colors to form a six-cube train, and then compare their solutions. One student might create a train with two red cubes and four yellow cubes, while another selects three red and three yellow cubes, and still another uses one yellow and five red cubes. Discussing the combinations that can be used to model any given number helps students recognize that a whole can be composed of various parts, and understanding part-whole relationships is the basis of addition. These discussions also build a foundation for understanding subtraction. If the train has six cubes, and they know that four of them are yellow and the rest are red, then students can count to determine that there are two red cubes. Decomposing numbers prepares them for later work with subtraction.

Once students are proficient at counting, and have mastered the numerals used to represent beginning numbers, they are ready to begin more formal addition. We can introduce formal notation and model how to write a complete mathematical equation by saying something like, "Melissa used five yellow cubes and then added one red cube to make her train have six cubes. Here is a way we can write that: 5 + 1 = 6." Students need hands-on experience using multiple examples and a variety of objects and contexts in order to develop a robust understanding of addition. For another example of basic addition, the teacher might begin at the concrete level by asking students to place goldfish crackers on a picture of a fishbowl, and then model how to record the results as a number sentence. For example, "Darius put three orange fish in his bowl, and then he added two red fish. I wonder how many fish are in his bowl now. Darius, can you count them for us? … Darius says he has five fish. Here is a way we can use numbers to show his fish: 3 + 2 = 5." At the pictorial level, students can draw pictures of fish to illustrate their concrete models, or use circles or tally marks, and then record their work by writing the number sentence. It is important to begin with concrete representation, pair the concrete objects with pictures, tally marks and other visual representations, and then pair these visual models with abstract numbers and symbols. (This introductory lesson using goldfish crackers and explicitly connecting CPA is available online.) Note that in the examples described here, the teacher included the abstract symbols from the beginning. All three forms of representation can be used together in an introductory lesson, but that does not mean that after initial instruction, students will be ready to eliminate concrete models. Textbooks designed for core instruction often begin with a concrete example, but then quickly progress to showing only pictures or only numbers within the first lesson. Research suggests that students with learning disabilities will typically need at least three experiences using concrete representation to model simple addition problems before they are ready to discard manipulatives and work solely with pictures and numbers, and three more experiences where visual representation continues to be included before they will be ready to rely only on abstract words and symbols. Therefore, the pace of instruction in core curricula may cause some students to become overwhelmed. While some students will progress rapidly through the CPA sequence, others will need more extended practice at each level. When a student can solve a simple problem independently and also explain what was done and why it was done that way, then that individual is ready to fade the supports and work with problems that only use abstract representation.

When students have a robust understanding of addition, they know that addition means combining parts. The relationship between the parts and the whole in an addition problem

Figure 8.1 Tape Diagrams and Bar Models



is clearly illustrated with bar models or tape diagrams. These schematic diagrams are frequently used when teaching students to solve word problems. We recommend also using them when introducing addition and subtraction. See Figure 8.1.

To create a tape diagram, draw a rectangle to represent the whole or total quantity (i.e. the sum). Then draw vertical lines within the rectangle to represent the parts. For example, to represent the problem 3 + 2 = 5, which is shown in the first drawing in Figure 8.1, the whole bar is labeled "5." The bar is divided into two parts because there are two addends, 3 and 2. One addend is written in each resulting box, so the diagram is an excellent way to model the *part, part, whole* relationship. In this example, 3 and 2 are the parts, and the whole is 5. When there are three addends, the bar is divided into 3 parts, as illustrated by the second drawing in Figure 8.1. This illustration shows that 3 + 2 + 4 = 9. Bar models are similar to tape diagrams, except they show the whole as one box, and the parts within another box that is drawn directly above or below the first. See the bottom examples in Figure 8.1. The tape diagram emphasizes how the parts combine to form the whole, while a bar model emphasizes the equivalence between the value of the parts and the value of the whole. Different math programs use different versions of these diagrams. Both are effective. During interventions, we recommend using the format used in the schools' core materials. During initial instruction, students can place actual objects on the model to illustrate a number sentence. Later, they can draw pictures in the boxes to show the addends, and then progress to writing the appropriate numerals next to their drawings. Finally, they progress to using diagrams that contain only numerals. Connecting these schematic drawings with a written number sentences helps solidify conceptual understanding.

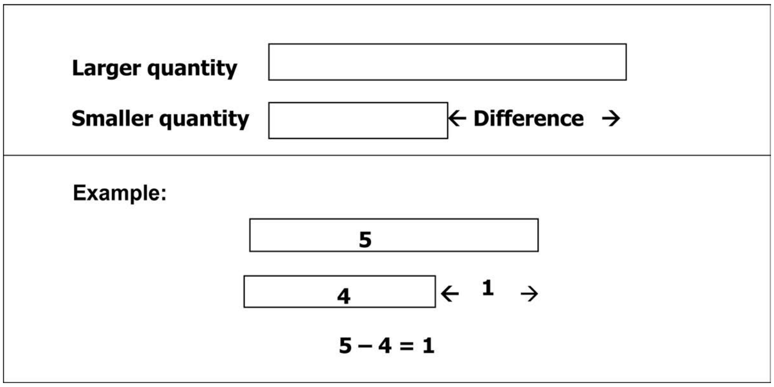
The process described here for introducing addition shares many similarities to the process found in most core math programs, but for students who struggle with mathematics, interventionists must provide additional supports. Supports are built into programs that have been validated for use in interventions, but if other materials are used, then additional intensification will be required. The ideas described above for increasing concrete and pictorial representation, and explicitly connecting the representations, are adaptations that increase support. Another way to intensify instruction is to make lessons more systematic and explicit, as explained in Chapter 5. Systematic instruction includes teaching pre-requisite skills to mastery before moving on to more advanced skills. It also means introducing new content in a carefully sequenced progression and providing ample opportunities for practice before introducing new content. Another way to intensify instruction is to emphasize and repeat precise, simple academic language (Powell & Fuchs, 2015). When introducing addition, emphasize terms like *part* and *whole* and *add* or *join* throughout the lesson. Emerging research suggests that adding gestures to our verbal explanations can improve understanding and retention (Goldin-Meadow & Alibali, 2013; Hord, Marita, Walsh, Tomaro, Gordon, & Saldanha, 2016; Walsh & Hord, 2019). To illustrate addition, you can put the items from one part in your right hand and extend it in front of you as you say "part." Then put the items from the other part in your left hand and extend it in front of you as you say "part." Finally, bring your two hands together, cupping the combined parts together as you say, "whole." Initially this is done with actual objects in your hands so the students can see a concrete example of how the parts combine to form the whole. Eventually, the same gestures can be used without actual objects. Teaching students to use the hand gestures and say "part, part, whole" when they are adding parts to form the whole can improve students' understanding of the addition process.

Once students can successfully combine two addends and explain the process, they can progress to adding three whole numbers whose sum is less than or equal to 20, and to learning strategies for solving addition fact problems such as 'counting on' or using combinations that equal ten. For example, if the child knows that 8 + 2 = 10, she can use that knowledge to solve the problem "8 + 3" by first solving "8 + 2" and then adding on one more (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). In Chapter 10, we will provide a detailed description of multiple strategies for solving basic math fact problems.

When students have mastered basic addition, they are ready for subtraction. Subtraction is the inverse of addition. Instead of joining groups, as in addition, subtraction involves separating groups. Students can separate a train of pop cubes by removing blocks of one color, and counting how many are left. The lesson that used goldfish crackers to introduce addition problems could easily be adapted to introduce formal notation for subtraction. Instead of adding goldfish crackers, we could begin the story with a group of goldfish in the bowl, and then have some fish "swim away" and count the number that remain. To model subtraction with gestures, reverse the process used for addition. Begin by cupping your hands together in front of you and say "whole." Then remove one part as you say, "part," while moving that hand away. Finally, extend your other hand forward as you say, "part that's left" or "remainder." The tape diagrams and bar models we introduced for addition also effectively model subtraction. Instead of combining parts to form the whole, students can begin with the whole, and then remove a part to find the remainder. Using gestures with the schematic diagrams helps reinforce understanding.

The subtraction problems described above all involved separating the whole into parts, either by decomposing a whole into its component parts, as in the pop train example, or through change over time, as in the goldfish example. Subtraction is also used to compare two groups. For example, we could ask students to each build a train using a handful of cubes, and then compare the number of cubes used they chose to use in their individual trains with the trains their classmates created. We could also put several different colored

Figure 8.2 Tape Diagrams for *Compare* Problems



fish in the "goldfish bowl," and then compare the number of red fish to purple fish, or the number of yellow fish to the number of orange fish. Such conversations provide an excellent opportunity to practice vocabulary such as *more/less*, *bigger/smaller*, and *difference*. Compare problems are modeled with a different tape diagram than the part/whole diagram we introduced preciously. Because comparison problems compare two different quantities, we use two separate bars to model comparison problems, as shown in Figure 8.2.

The gestures used to illustrate compare problems are also different than those used in part/whole problems. To illustrate the difference between two sets with gestures, place your hands on top of each other in front of your chest, with one palm resting on the back of the other hand, and then raise the top hand up and down as you describe the sets. The top hand represents the larger quantity, while the bottom hand represents the smaller quantity. The space created as your hands move up and down represents the difference. (These are the same gestures used to introduce magnitude comparison in the previous chapter.) Note that subtraction problems that involve separating groups differ significantly from subtraction problems that involve comparing two sets of objects, and we have suggested using different gestures to illustrate each type of subtraction problem. Core materials frequently mix the two types of problems within a single lesson. Again, it is important to follow the principles of systematic instruction during interventions. Introduce these two forms of subtraction in different lessons, while highlighting how the two formats are the same and how they are different.

When modeling subtraction problems, remind students to only display the minuend. In other words, they should use objects to show the top number if the problem is written vertically or the first number when the problem is written horizontally. When we introduced addition, we modeled both parts and then combined those parts to find the total. In a subtraction problem, we only model the total, and then remove or cross some out and count to determine the difference. If students use counters to model both the numbers in a subtraction problem, they have actually illustrated an addition problem. See Figure 8.3.

Emphasizing the relationship between addition and subtraction can help students understand and solve subtraction problems. For example, if students see a pile of three counters and another pile of five counters, they can determine that there are eight counters in all. If we cover the pile of five counters and leave the rest exposed, and then ask students to determine how many counters are hidden, we have created a subtraction problem: $8 - 5 = 3$. Although we can solve this problem by beginning at eight and counting down, children often approach this as an addition problem and count up from

Figure 8.3 Modeling Subtraction

Important Terms

- Minuend: Original quantity from which an amount will be subtracted.
- Subtrahend: Quantity to be removed.
- Difference: The quantity remaining after subtraction (answer).

8 ← minuend
- 5 ← subtrahend
3 ← difference

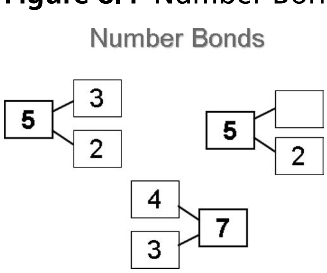
Representing a Subtraction Problem

1. Model the minuend.
2. Remove or cross out the number of items stated in the subtrahend.
3. What is left is the difference.

five to eight. Counting up, or "counting on," can be a very effective strategy for solving simple subtraction problems, because it builds on students' previous experiences with addition. When students recognize that they can count up or count down to obtain the solution, they begin to understand the relationship between addition and subtraction, which enhances their ability to reason numerically. However, we need to use caution here. When different approaches are introduced simultaneously or in quick succession, without providing enough practice for students to solidify their understanding of one concept before the next one is introduced, students can become confused. Some students will end up jumbling addition and subtraction if they are offered the option of either counting up or counting down, and these students often struggle when asked to write an equation to represent the problem. Again, we can minimize problems if we follow the principles of systematic instruction during interventions and introduce content in carefully sequenced chunks.

At the visual level, the connection between addition and subtraction is graphically illustrated by the "number bonds" used in Singapore Math. Singapore's consistently excellent results on the Trends in International Mathematics and Science Study (TIMSS) place it among the best in the world in mathematics achievement. The number bonds they use to illustrate part-whole relationships provide a clear visual representation of the connection among the numbers in a fact family, as shown in Figure 8.4.

Figure 8.4 Number Bonds



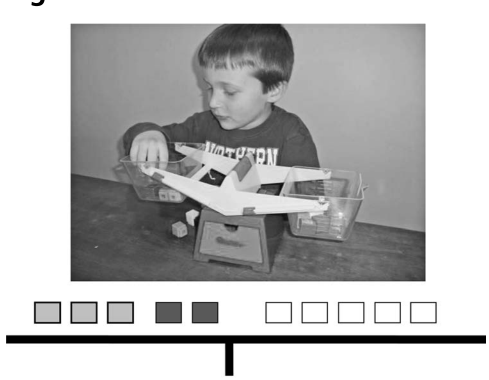
Students can use the number bonds to determine that 6 + 2 = 8 and 2 + 6 = 8, and also that 8 – 2 = 6 and 8 – 6 = 2. Composing and decomposing numbers this way increases students' number sense and helps develop computational fluency. Note that number bonds are a form of two-dimensional diagram. They cannot replace concrete representation, and should only be introduced after students have had sufficient experiences at the concrete level. They are more abstract than bar models and tape diagrams, and so should be introduced after the students are comfortable with tape diagrams or bar models. Some popular core materials introduce number bonds early in first grade. While this may be appropriate for students who are performing at grade level, students who struggle may need extensive practice with concrete representation, and with bar models or tape diagrams, before they progress to the more abstract representation provided by number bonds.

Another way to model the relationship between addition and subtraction is with dominoes. Counting the dots on each side of the domino creates an addition problem, while hiding one side illustrates the related subtraction problem. Dominoes also provide an excellent example of the commutative property of addition. For example, a domino with 5 dots on the left and 3 dots on the right represents the addition fact: 5 + 3 = 8. When flipped, it shows the related fact: 3 + 5 = 8. The total number of dots on the domino does not change, and so students can clearly see the commutative property of addition, i.e. that changing the order of the addends does not affect the sum.

Whether we are working with addition or subtraction, when we introduce operations, we need to make sure students accurately understand the meaning of the equal sign used in formal notation. Equality is a relationship, not an operation. The "equals" sign is the mathematical symbol placed between objects, numbers, or sets that have the same value, but when students first encounter this symbol, they often interpret it to mean, "find the answer." This misinterpretation will cause difficulty when learners are asked to tackle equations presented in an unfamiliar order, such as "9 = 4 + 5" or "12 – ? = 8." Understanding equivalence is also essential for students' later work with algebraic equations. To help develop this important concept, we can model it concretely with a pan balance. In Figure 8.5, we show a student adding more counters to one of the pans in order to make the scales level.

Providing multiple opportunities for students to use a balance to model equivalence will help them understand the concept of equality. The pan balance also provides an effective model for solving problems with missing middle addends (e.g. 2 + ? = 5), which students often find especially challenging. A simple line drawing of a pan balance, like the one shown at the bottom of Figure 8.5, will help students make the transition from concrete to visual

Figure 8.5 Pan Balance



3 + 2 = 5

representation. Drawing an "equals" sign below the balance connects the abstract symbol to the visual model. Explicitly linking the concrete and visual representations with the abstract symbol will help students understand that the set on the left of the equal sign must have the same value as the set on the right. A variety of sites offer virtual pan balances (see, for example, https://www.nctm.org/Classroom-Resources/Illuminations/Interactives/Pan-Balance-Numbers/), which can provide an excellent alternative for remote learning experiences, for students practicing at home with parents, or for working with older students who may view an actual balance as "babyish." Working with pan balances is valuable in core instruction, and is even more important during Tier 2 and Tier 3 interventions. Students who have a robust understanding of equivalence will be much better prepared to tackle algebraic equations in the upper grades.

As we discussed previously, numerical understanding involves the ability to use multiple models to represent the same problem or procedure. We therefore need to provide opportunities for students to use a variety of manipulatives to model each skill, and make sure they can transition fluidly among the different forms of representation. Given a number sentence, can the student model it with objects or pictures? Given a concrete model of an addition or subtraction problem, can he draw pictures or use tallies to model the same problem? Can she write the number sentence and suggest a story to go with the model? Note that the problems provided at this level are very simple addition problems designed to help students develop the basic concept that addition involves joining two or more sets. We use story problems because students will attend and retain information better when they perceive it as meaningful and relevant (Archer & Hughes, 2011; Wolfe, 2010). However, many students who struggle with mathematics have extreme difficulty solving story problems due to language deficits that interfere with their ability to process the problem. When introducing addition and subtraction, the purpose of the story problems is to provide a context for the operation, so the problems selected should be simple and straightforward. We do not want the child frustrated by the language of the problem or confused by complex distractors. Strategies for teaching problem-solving and dealing with more complex problem scenarios will be discussed much more thoroughly in Chapter 12.

Developing Computational Fluency with Basic Facts

Computational fluency is the ability to compute accurately, quickly, and effortlessly. Therefore, the IES Practice Guide identifies developing computational fluency as a priority for students receiving interventions (Gersten et al., 2009). We devote an entire chapter to strategies to help students compute accurately, quickly, and effortlessly. While automaticity with basic facts facilitates mathematical progress, students can continue to engage in mathematical problem-solving activities and build conceptual understanding of operations long before they master the basic facts. In the rest of this chapter, we discuss procedures for enhancing students' conceptual understanding of addition and subtraction of larger numbers, as well as developing concepts and strategies for multiplication and division. We recommend that the topics that follow be addressed concurrently with the ongoing practice with basic facts which will be discussed in Chapter 10.

Solving Multi-Digit Addition and Subtraction Problems

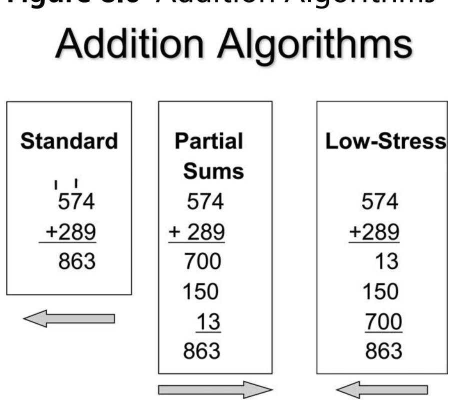
Once students understand place value and can solve addition and subtraction fact problems, they are ready to tackle problems involving larger numbers. Typically, students first add a one-digit number to a two-digit number and subtract a one-digit number from a two-digit number. Later they add and subtract two, two-digit numbers, first without regrouping, then with regrouping, and eventually progress to solving problems that contain larger numbers. The Common Core Standards stipulate that by the end of second grade, students should be able to perform the following addition and subtraction skills:

> CCSSM.2.NBT.7: Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010)

Students can begin working with multi-digit addition and subtraction at the concrete level by first modeling a problem with concrete objects, and then counting to determine the answer. This is an excellent introductory activity because it develops initial understanding without requiring any written computation. Eventually, however, students must learn to use an algorithm so they can solve these multi-digit problems without the aid of concrete or pictorial representation. An algorithm is a set of step-by-step procedures for solving a problem, and there is more than one algorithm that leads to the correct solution of any problem. Historically, students were taught only one way to add or subtract multi-digit numbers, i.e., working from right to left and "carrying" or "borrowing" (now called *regrouping*). This is the *standard algorithm*, and it is the way most teachers learned to add and subtract. Students sometimes struggle with this algorithm because it approaches problems in a very piecemeal fashion. For example, a seven in the tens column is really seven tens, or 70, while a seven in the hundreds column represents 700. In the standard algorithm, digits are manipulated in a discrete fashion that may cause students to forget their place value, resulting in some of the errors frequently observed among students who struggle with regrouping. Students may also confuse the right to left progression required to correctly execute the standard algorithm with the left to right progression used in reading. In an effort to prevent some of these problems, many math programs now teach students to use an alternative algorithm before they introduce the standard algorithm. Figure 8.6 provides an example of three different algorithms for solving addition problems.

In the rest of this chapter, we suggest ways to teach each of these algorithms, along with suggestions for introducing algorithms for the other operations with whole numbers. These algorithms are taught in both core instruction and tiered interventions. When interventionists introduce any mathematical algorithm, it is important to follow the recommendations

Figure 8.6 Addition Algorithms



for intensifying instruction, including (1) using systematic, explicit instruction, (2) giving students a list of steps to follow, (3) explicitly modeling how to use the algorithm, (4) selecting visual representations that match the algorithm being used, and then explicitly connecting the visual representation to the abstract representation, (5) using precise academic language and adding gestures for emphasis, and (6) having students explain their reasoning. We provide examples of how to incorporate these recommendations to intensify instruction when introducing each of these algorithms.

The Low Stress or Partial Sums Algorithms for Addition

To introduce the low-stress or partial sums algorithm, begin at the concrete level by having students use blocks to represent each addend, before introducing the abstract algorithm. Allowing students to manipulate the blocks reinforces the place value of each digit in the problem. In addition, teachers need to be careful to use language that reflects place value of the digits, so that students maintain conceptual understanding of what they are doing. For example, instead of referring to a 7 in the tens column as "seven," we should call it "seven tens" or "seventy," and encourage students to do the same. The alternative algorithms allow students to work from right to left or left to right. To solve the problem from left to right, students begin by modeling the largest place value digit. In the example of the Partial Sums Algorithm in Figure 8.4, that would mean using base-ten blocks to first model all the hundreds, and then adding them up and recording the total value ("5 hundreds plus 2 hundreds is the same as 7 hundreds," or "500 plus 200 equals 700, so we will write 700 below the line"). Next, students would move to the tens column to model, add, and record the value of tens in the problem ("7 tens plus 8 tens is the same as 15 tens, or 150. Let's record 150 here."). Finally, they would do the same with the numbers in the ones column (9 plus 4 = 13). To find the solution to the full problem, simply add the partial sums, again going from left to right. In this example, we see a total of 8 hundreds, 6 tens, and 3 ones, so the solution is 863. The same problem can also be worked from right to left by first adding and recording all the ones, then the tens, and then the hundreds, and finally combining these subtotals to determine the grand total. In both of these alternative algorithms, no regrouping is required. However, in order for students to use this algorithm successfully, they must first have developed a solid understanding of place value.

To support executive functioning and self-regulation, provide students with a list of steps to follow, such as those shown in Figure 8.7. Give them a laminated copy of the steps, and have them check off each step as they follow it. This helps them monitor their performance, and also provides a useful tool when they complete problems independently or when they are in the general education classroom away from the guidance of the interventionist. Self-monitoring is an evidence-based practice that has been shown to increase achievement and engagement, and so provides an effective way to intensify

Figure 8.7 Steps for Using the Partial Sums Algorithm for Addition

Steps for Using the Partial Sums Algorithm for Addition
☐ Add the hundreds, and record. ☐ Add the tens, and record. ☐ Add the ones, and record. ☐ Add the totals together, and record. ☐ Check. Does my answer make sense?

instruction (The IRIS Center, 2020). Some teachers have been told that giving students a list of steps to follow makes the process too procedural; in other words, students may solve the problem by rote without truly understanding what they are doing. This is a valid concern if the teacher simply tells the student, "Do this, then do this, then do this." Instead, it is important to make sure that the student fully understands the meaning that underlies each step in the process. When you first model the process, use "think-alouds" to model your thinking and help students understand the "why" behind each step. Then highlight the connection between the concrete representation and the abstract numbers by having students model a column, add and record the results, and then explain what they did and why they did it that way. Having students explain their reasoning, and critique the reasoning of others, helps develop deep understanding.

These alternative algorithms are simple and effective, and for students who are functioning far below grade level, the team may elect to teach these algorithms exclusively. Students working in the core curriculum are expected to master multiple algorithms, but for students who are behind academically, the time required to master additional algorithms for addition might be more profitably devoted to working on different skills. If the decision is made to introduce multiple algorithms, allow students sufficient time to master one algorithm and solidify their understanding before introducing a second algorithm.

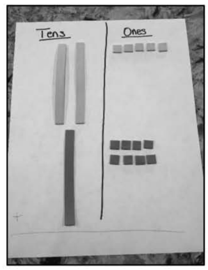
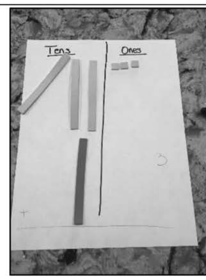
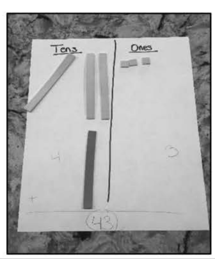
The Standard Algorithm for Addition

When we introduce the standard algorithm, we once again begin at the concrete level. However, because the steps in the standard algorithm differ from the steps used for the alternative algorithms, the way we model the problem will be different. Ten-frames, base-ten blocks, KP Tiles, and DigiBlocks are all tools for modeling the standard algorithm. Select a manipulative that students have used before in order to help connect the standard algorithm to their previous experiences with place value. The purpose of using concrete representation is to give meaning to the abstract algorithm, but this goal can only be accomplished when we explicitly connect each step in the algorithm with the concrete manipulation. The models suggested for partial sums and low stress algorithms do not adequately reflect the regrouping process, so we will describe ways to model the standard algorithm. It is helpful to introduce the standard algorithm using problems that do not require regrouping, so that students first master the idea of beginning in the ones column, and later learn the process of regrouping. After students can successfully execute the standard algorithm to solve problems that do not require regrouping, then they are ready to tackle regrouping problems. Because regrouping requires a solid understanding of place value, it is beneficial to review expanded notation and the "Making Trades" game that was described in the online resources for Chapter 7. Use the same language of "making trades" to introduce regrouping in the addition algorithm.

When introducing the algorithm, again it is helpful to give students a list of steps to follow. Teaching them to monitor themselves by following written steps supports executive functioning and self-regulation, and facilitates their ability to solve these problems independently. To model solving a problem using the standard algorithm, begin in the ones column. Students should use blocks to model the values in the ones column, and then record the total in the abstract problem, so they see the relationship between the concrete representation and the paper-and-pencil activity. Next, they can model the tens and record the result.

Figure 8.8 shows an example of a teacher demonstrating how to use the standard algorithm to add two, two-digit numbers. Notice how the teacher shows the steps, the

Figure 8.8 The Standard Algorithm for Addition

The Standard Algorithm for Addition: 25 + 18			
Strategy Steps	**Abstract**	**Concrete Model**	**Scripted Explanation**
1.) Model. Tens / Ones 2 / 5 + 1 / 8	25 + 18		Step 1 says model. First I will show the twenty-five with base ten blocks. The 2 is in the tens place so I need 2 tens blocks. I'll get those and place them under the tens column. Then I need 5 ones in the ones column. Next, I will show 18 with the base ten blocks. There is a one in the tens place, so I need to put one tens block in the tens column. Then I need 8 ones in the ones column. OK. I have shown the problem 25+18 with my base ten blocks. I can check off Step 1.
2.) Add the ones. Ten or more? Make a trade. Record.	1 25 + 18 3		Now we are going to add the ones column. Remember if we have more than ten in this column, then we need to make a trade. 5 plus 8 is 13. Uh oh, I have more than ten. Let's make a trade. I'm going to trade ten ones for one ten. I'll put that ten in the ones column, here. Now let's count how many we have in the ones column. 1, 2, 3. We have 3 ones. I will record 3 in the ones column of my problem. I also need to record the ten that I traded. Ok. Let's check off Step 2.
3.) Add the tens. Record.	1 25 + 18 4 3		"Now let's add the tens. 1, 2, 3, 4 tens. We can record 4 on the tens column. We now have 4 in the tens column and 3 in the ones column, so our answer is 43."
4.) Check			Let's check. I'm going to count all my blocks. I have ten, twenty, thirty, forty, forty-one, forty-two, forty-three. It matches!

abstract problem, and the concrete model side-by-side, and explicitly links the concrete and abstract representations at *each* step. Explicitly linking the various representations builds a more robust understanding of the meaning behind the abstract algorithm. Breaking the procedure into small steps, giving the students a list of steps to follow, modeling how to apply the steps to the problem, and explicitly connecting the various representations, are all examples of effective ways to intensify instruction for students during interventions.

In this example, the teacher did all the talking. Because using base-ten blocks to model a two-digit number is a skill that students should have already mastered before the standard algorithm is introduced, the teacher could have asked for student input when executing Step 1.

Steps 2 and 3 are introducing new content, so it is better to model this step directly, rather than ask for student input. The teacher uses the think-aloud strategy to model her reasoning when executing these steps. In Step 4, she checked the answer by counting all the base-ten blocks. Since counting base-ten blocks is a skill that students should have already mastered before the teacher introduces the standard algorithm, this would be another place where the teacher could have encouraged student input and discussion. See the online resources for a more detailed example of the modeling portion of a lesson introducing regrouping with the standard algorithm.

When students can explain the procedure using concrete manipulatives, the blocks can be faded, and students can use drawings to model their work. Once they can effectively explain the procedure using the visual representations, they are ready to solve addition problems using just the abstract numbers.

The Standard Algorithm for Subtraction

The procedure for introducing multi-digit subtraction is similar to that suggested for introducing addition. We recommend using the standard algorithm for subtraction, because learners with a history of mathematical difficulty often have deficits in working memory that make alternative subtraction algorithms especially challenging.

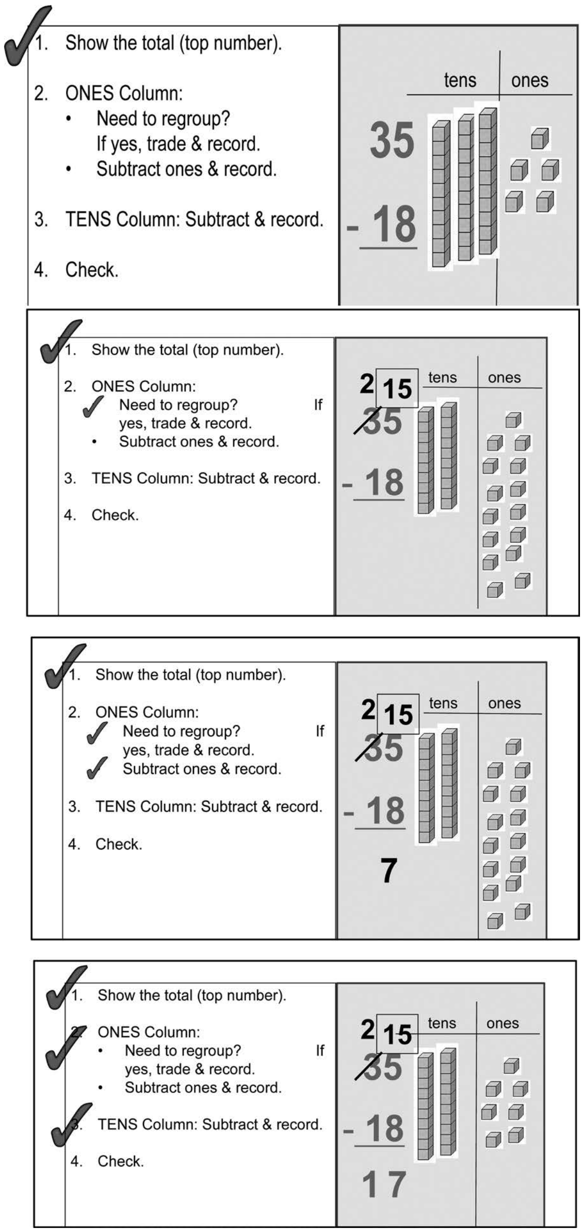
Before introducing problems that require regrouping, have students again play Making Trades, the game that was described in the online resources. However, in order to model the regrouping process in subtraction, students should start with a flat and remove blocks until they have none left. The decision about whether to use ten-frames, base-ten blocks, DigiBlocks, or some other manipulative should be based on which manipulative will most effectively help students connect the new procedure with their existing understanding of place value. Remind students that, when modeling subtraction problems, we lay out blocks to represent only the minuend, or top number in the problem, because if we laid out both the minuend and subtrahend, we would actually be modeling an addition problem.

To introduce the standard algorithm, first give students the steps and allow them to solve problems using base-ten blocks, but without introducing the abstract notation. Next, solve the problem again, and this time pair each step in the abstract problem with the concrete representation. Linking concrete and abstract representation helps students understand the rationale for each action. See the example in Figure 8.9.

The online resources contain three examples of teachers modeling the standard algorithm for subtraction. First, in the resources for Chapter 5, the example we provided of how to model an explicit strategy involved the standard algorithm for regrouping in subtraction. In the online resources for this chapter, there is a similar example using base-ten blocks to introduce the standard algorithm, as well as an example of a teacher using ten frames to model subtraction. In each lesson, the teacher explicitly connects the various forms of representation so that students can see the purpose of each step in the algorithm.

Because subtracting across zeroes requires special procedures that can confuse students, it is best to avoid zeroes during students' initial experiences with regrouping. Once students can execute the standard algorithm independently using concrete, pictorial, and abstract representation, and can explain what they are doing and why they are doing it, then problems with zeroes can be systematically introduced. After the regrouping algorithm has been introduced, some students overgeneralize and try to regroup in every column of every problem, whether it is appropriate or not. Mixing problems so that some require regrouping and others do not, and some require trades only in the ones column and others only in the tens, will encourage students to think more carefully about what they are doing. They

Figure 8.9 Modeling the Standard Algorithm for Subtraction



need practice deciding whether to regroup or not, and they need practice explaining and justifying their decisions. When students can explain what they are doing with the models, then the teacher can fade the concrete supports and have students work exclusively with abstract numbers.

Intensifying Instruction During Interventions

Although the process for teaching addition and subtraction to students who receive tiered supports is similar to instructional strategies presented in core (Tier 1) instruction, there are important differences. Many educators who provide math interventions do not have access to a validated program where intensive intervention practices are already built into the program. Others work with students who require even more individualized supports. Ideas for intensifying instruction to meet the needs of learners receiving tiered support were discussed throughout this chapter. Here is a summary of some of the many ways to intensify instruction during interventions.

1. Use systematic instruction. Select objectives carefully. Sequence them from easiest to hardest, and make sure that pre-requisite skills are mastered before introducing more complex content. If students struggle, objectives can be further broken down into component parts or steps. If a student struggles to complete all the steps in a single lesson, then the lesson could be broken down to focus on only one or two steps each day. Although it will take longer to introduce the complete procedure, this approach often saves time in the long run because it reduces the need for reteaching. To avoid overwhelming students' cognitive capacity, pace instruction so that students solidify their understanding of one concept or skill before introducing another.
2. Use explicit instruction. Follow the guidelines described in Chapter 5. If the available materials do not use this high-leverage practice, then modify the lesson to include all the elements of explicit instruction.
3. Give students a written list of steps to follow, and teach them to refer to the list as they work. Many students who struggle with mathematics have deficits in executive functioning. Teaching them to monitor their progress by checking off steps has been shown to increase achievement.
4. Follow the CPA continuum. Always begin at the concrete level, and allow students sufficient time exploring and mastering math concepts with manipulatives and pictorial representation before expecting them to solve problems using only abstract words and numbers. Explicitly connect the concrete and pictorial representations to the abstract algorithm to build deep understanding. When students can explain the meaning of each step, then they are ready for interventionists to fade the concrete and visual supports and focus on developing procedural fluency with abstract representation.
5. Use precise academic language when you model mathematical procedures. Emphasize vocabulary in each lesson, and have students practice using the academic vocabulary themselves. Supplementing verbal language with gestures has also been shown to increase understanding and retention for some students.
6. Have students explain what they are doing, and why they are doing it this way. Asking students to explain their reasoning helps them solidify understanding, and also provides valuable formative assessment information that can be used to refine instruction. Core curriculum materials increasingly stress the importance of communication in mathematics. Too often students receiving interventions have learned to use tricks and follow steps by rote, without developing conceptual understanding. Asking students to explain their own reasoning, and to understand and critique the reasoning of others, is important to develop mathematical proficiency.

Summary

Research studies have documented the value of using explicit instruction and following the CPA continuum when introducing numbers and operations. According to the IES Practice Guide, "A major goal of interventions should be to systematically teach students how to develop visual representations and how to transition these representations to standard symbolic representations used in problem solving" (Gersten et al., 2009). In other words, we need to use the concrete and pictorial representation initially to help students develop conceptual and procedural understanding, but we must carefully link these representations to standard abstract notation and then systemically fade the supports and allow students to become proficient in solving problems using standard symbolic representation.

In this chapter, we provided suggestions for developing students' conceptual understanding by using explicit strategies and systematically linking concrete and visual representations to the abstract algorithms used when computing whole numbers. In the next chapter, we focus on operations with multiplication and division. Developing computational fluency with basic facts for all four operations is discussed in Chapter 10.

9

Operations with Whole Numbers: Multiplication and Division

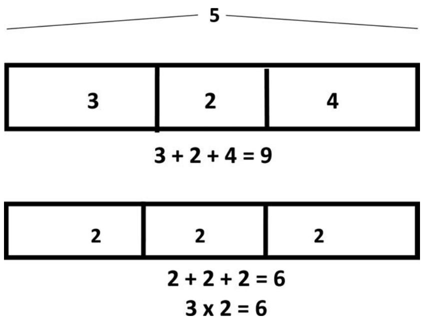
Number sense and operations with whole numbers form the foundation for all higher mathematics. The IES Practice Guide recommends that "instructional materials for students receiving interventions should focus intensely on in-depth treatment of whole numbers in kindergarten through grade 5" (Gersten et al., 2009, p. 6). In Chapter 7, we focused on developing number sense, and in Chapter 8, we addressed addition and subtraction of whole numbers. In this chapter, we focus on multiplication and division of whole numbers, The Common Core State Standards lay the foundation for multiplication in second grade, and it is a major focus of third grade mathematics. By the end of third grade, students should understand the concept of multiplication and division and have strategies for multiplying and dividing within 100. By the end of fourth grade, they should be fluent with multiplication and division facts and multi-digit multiplication, and begin developing an understanding of division using multi-digit dividends. Fluency with all whole number operations is expected by the end of sixth grade (National Governors Association, 2010).

Developing Conceptual Understanding of Multiplication

Multiplication is an extension of addition, and so students should master the concept of addition before multiplication is introduced. One of the simplest ways to show the relationship between addition and multiplication is with bar models or tape diagrams. In Chapter 8, we described how tape diagrams show the *part/whole* relationships in addition. The same diagrams also illustrate *part/whole* relationships in multiplication. See Figure 9.1

When the parts are of different sizes, as they are in the first drawing in Figure 9.1, we add to find the sum. When all the addends are of equal value, as they are in the second example, we can still solve the problem by adding, but we can also solve the problem by multiplying the number of parts times the value of each part. Either approach provides the correct answer, but multiplication is more efficient. When we introduce multiplication, we want to make sure that students recognize the relationship between multiplication and addition. Multiplication is not some new and strange process. It is simply a more efficient way to solve problems when there are multiple parts that are all the same size. When tape diagrams are used to model multiplication facts, the first factor determines the number of parts. The second factor is written inside each box to show the size of each part, as shown

Figure 9.1 Tape Diagrams for Addition and Multiplication



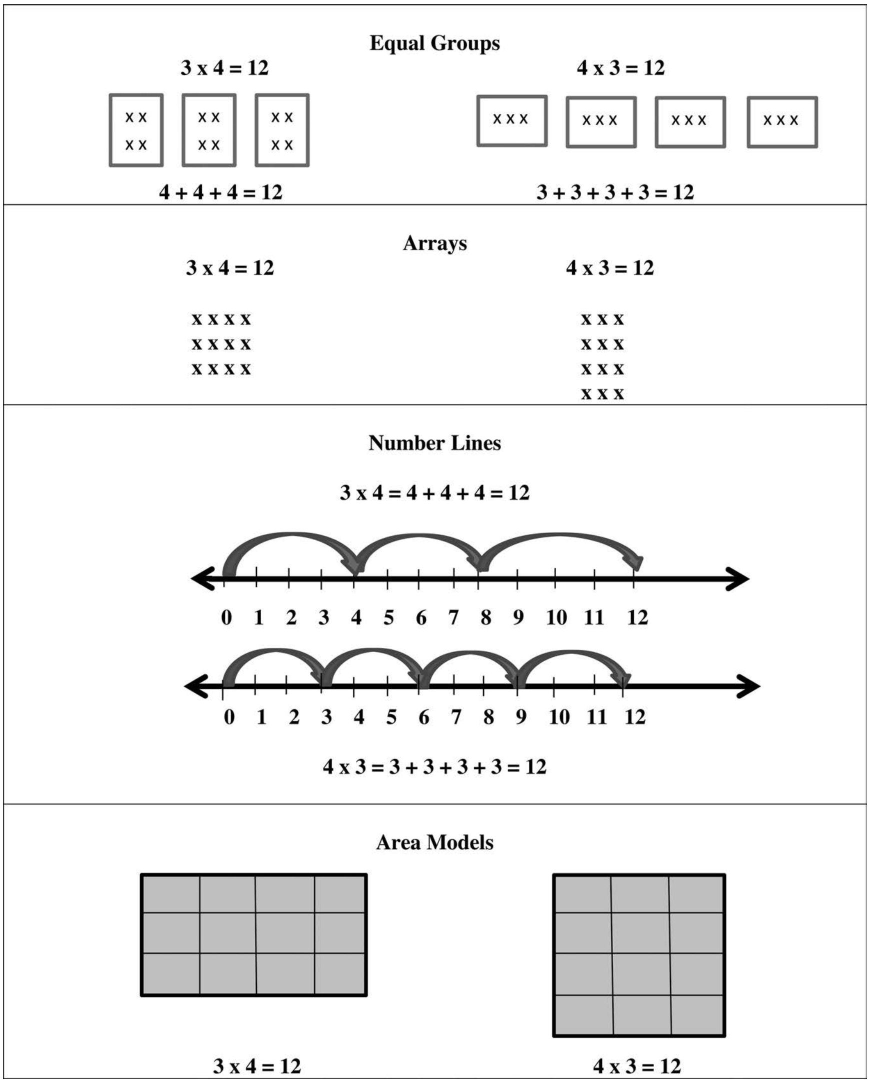
in Figure 9.1. The multiplication fact 3 × 2 is *not* written by placing a 3 in one box and a 2 in the other box, because that would model an addition fact. Students frequently make this error when they first learn to use tape diagrams to model multiplication. The number bonds described in Chapter 7 do use one factor in each box, but tape diagrams show the relative size of the parts, not just abstract numbers. Because bar models and tape diagrams are less abstract than number bonds, tape diagrams should be introduced first.

A variety of other models are frequently used to help students understand the multiplicative process, including equal groups, arrays, repeated addition illustrated with number lines, and area models. See Figure 9.2. The top illustration depicts equal groups. In a multiplication fact, the first factor is traditionally used to indicate the number of groups, while the second factor tells the number of members in each group. In this figure, we model 3 × 4 with three groups containing four objects per group. A simple way to help students understand the idea of equal groups is to give students paper plates to represent the groups, and let them place counters on the plates to show how many members are in each group. Coffee stirrers or craft sticks in paper cups can also help students develop a concrete understanding of basic multiplication. The commutative property tells us that the order of the factors does not affect the product; both 3 × 4 and 4 × 3 result in a product of twelve. However, order does make a difference when we model multiplication facts, as shown in Figure 9.2. When we represent 3 × 4, we draw three groups and put four members in each group, while the illustration of the related fact, 4 × 3, shows four groups with three members in each group. The product is the same, but the drawings look different. Students who are struggling to develop a basic conceptual understanding of multiplication will need to spend time exploring and discussing this relationship.

As students become more comfortable representing multiplication with equal groups, interventionists can enrich their conceptual understanding by introducing additional models. The second illustration in Figure 9.2 shows the same fact problem, 3 × 4, modeled with an array. In an array, factors are organized into rows and columns. We will use the first factor to designate the number of rows in the array, and the second factor to indicate the number of columns. Arrays provide a clear illustration of multiplication facts. They are also a simple way to introduce the commutative property, because by flipping the array, the same drawing illustrates the related fact. Arrays graphically show that reversing the order of the factors still yields the same product.

The third illustration in Figure 9.2 uses a number line to model multiplication as repeated addition. The multiplication fact, 3 × 4, is shown as three groups of four, or 4 + 4 + 4, and the related fact, 4 × 3 is modeled with four groups of three, or 3 + 3 + 3 + 3. In Chapter 7, we described how MathLine can be used to connect concrete, visual, and abstract models of numbers on a number line (see Fig. 7.4). MathLine is also an excellent tool to model

Figure 9.2 Representing Multiplication



multiplication facts on a number line. To show 3 × 4, students would begin with all the rings pushed to the right so that the zero is exposed, and then create three groups, each containing four rings. When students push all three groups to the left, they will see the product, twelve, displayed to the right of the rings.

The bottom illustration in Figure 9.2 shows an area model. An area model is similar to an array, except that it uses square units placed side by side so that they touch each other, rather than discrete objects, to form the rows and columns. In other words, in an array, there can be space around each object, but in an area model, all the squares touch each other, with no spaces between them. If you think about geometric area, the factors form the length and width of a rectangle, and the area within the perimeter of the rectangle represents the product. Area used to be a topic introduced as a discrete concept in geometry, but area models are now introduced with multiplication, and meaningfully connect multiplication and geometry.

Exposing students to multiple ways of modeling a problem helps develop robust understanding. Core materials often introduce a variety of representations in quick succession.

To intensify instruction for students who struggle with mathematics, we encourage interventionists to follow the principles of systematic instruction. Introduce one method at a time, and make sure students fully understand it before introducing a new approach. Being exposed to too many models in rapid succession can overload students' working memory and actually hinder learning. Students who require tiered supports benefit from learning about all of the different representations, but carefully pacing and sequencing the instruction will improve achievement outcomes.

Understanding Division

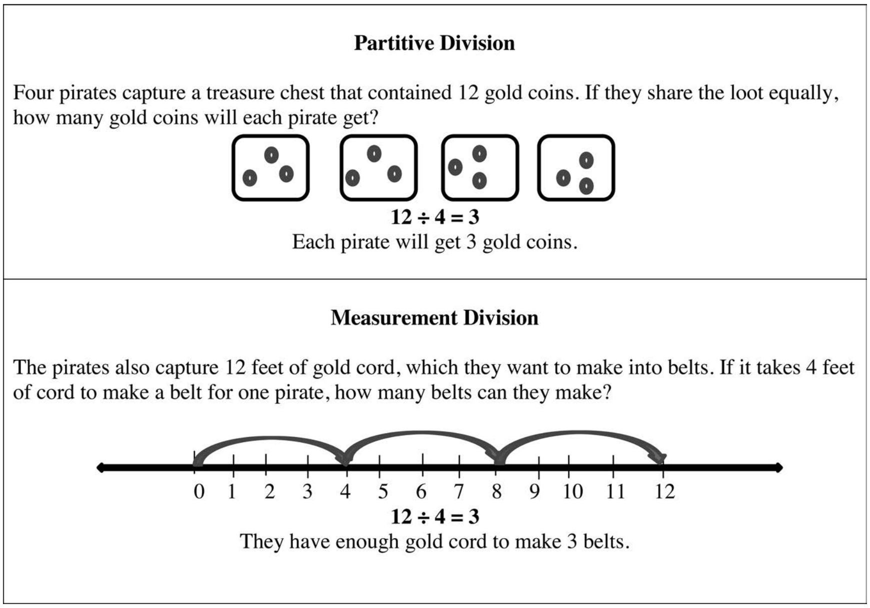
Division is the inverse of multiplication, and is typically introduced after students have had successful practice solving multiplication facts. When students understand the relationship between the two operations, they can use their knowledge of multiplication to solve division problems. Students can represent division problems with the same models they used previously for multiplication. Seeing how the models are connected will help them understand the relationship between the two operations. For example, we suggested modeling the multiplication problem of $3 \times 5 = 15$ by giving students three plates and having them place five counters on each plate to show the product, 15. To model the related division problem $15 \div 3$, give students 15 counters and three paper plates, and let them distribute the counters equally among the plates, and then discuss how the two operations are connected. Division can also be modeled as an array. To represent $15 \div 3$, distribute 15 counters evenly in three rows. The quotient is the number of columns created, i.e., five. The same division fact could also be shown using an area model by forming a rectangle containing 15 square tiles arranged in three rows. Again, the quotient is the number of columns created, i.e., five.

One significant difference between multiplication and division can confuse students. In multiplication, the order of the factors does not matter. The commutative property tells us that the product of $a \times b$ is the same as the product of $b \times a$. The same is not true in division. Order definitely matters; the quotient of $b \div a$ is not the same as the quotient of $a \div b$. Providing students with opportunities to use objects to model both problems, and then discuss the resulting solutions, helps students develop a robust understanding of division.

Division can be interpreted two different ways: as *partitive* division or as *measurement* division. Figure 9.3 shows examples of partitive and measurement division.

In partitive division, the divisor indicates the number of groups, and students must determine how many items are in each group. In the division example described above, students modeled $15 \div 3$ by distributing 15 counters equally onto three paper plates. That is an example of partitive division, because we know both the total number of objects and the number of groups, or parts, and are solving the problem to determine the number of items contained in each part. In measurement division, the divisor represents the size of each group or part, so students solve the problem to determine how many equal-sized pieces they can form. They can use the same materials described previously for partitive division, but instead of interpreting the divisor as the number of plates needed, they would use the divisor to determine the size of each group. To model $15 \div 3$ as a measurement problem, they would measure "divisor-sized" groups. In other words, they would place three counters on each plate until they run out of counters. They will find that it takes five plates to use all the counters. When we use repeated subtraction to solve a division problem, we are using measurement division. See the number line example in Figure 9.3. Although students need to be able to solve both partitive and measurement division problems, students who are easily confused or frustrated will benefit if they have the opportunity to become comfortable with one format before tackling problems involving the second type of division. Textbooks often mix

Figure 9.3 Partitive and Measurement Division



the two types of problems within a single lesson, so to intensify instruction, interventionists can introduce the two types of division separately, following the principles of systematic instruction.

Students need many experiences using concrete and visual models of multiplication and division to develop a sound conceptual understanding of these operations. When they can use models effectively and explain their work, then the concrete and visual supports can be faded and they can begin working with purely abstract representation.

Developing Fluency with Multiplication and Division Facts

Students who are mathematically proficient have a solid conceptual understanding of the operations, and they can also compute products and quotients quickly and easily. Automaticity with basic facts makes it easier to solve larger multiplication and division problems, because instead of having to think about the computation, students who have mastered the facts can devote their full attention to the problem-solving process. Computational fluency with multiplication and division facts is also beneficial later when students begin reducing fractions and determining common denominators, as well as when performing multiplication and division operations with fractions and decimals. For this reason, the IES Practice Guide recommends that interventionists devote about ten minutes of each intervention session to developing computational fluency, and then spend the rest of the period working on other skills (Gersten et al., 2009). We devote Chapter 10 to strategies for developing computational fluency with basic facts. However, lack of computational fluency does not preclude a student from beginning to work with multi-digit multiplication and division. Students who have not mastered the facts will need additional supports, such as a fact chart, to help with computation, but they can still begin learning multiplication and division algorithms while they continue to review basic facts. In the rest of this chapter, we discuss ways to teach and model algorithms for multi-digit multiplication and division.

Figure 9.4 Algorithms for Multiplication

Multiplication Algorithms			
Standard Algorithm	**Repeated Addition**	**Low Stress or Partial Products Algorithms**	
^{1} 12 x 7 8 4	12 x 7 12 12 12 12 12 12 + 12 84	12 x 7 14 70 84	12 x 7 70 14 84

Multi-Digit Multiplication Algorithms

When students participate in the core curriculum, they are exposed to several algorithms for solving multi-digit multiplication problems. See Figure 9.4 for examples of some of the possible multiplication algorithms. Students who require tiered support in mathematics need to be able to use at least one algorithm to solve multi-digit multiplication problems efficiently. Whether these students should learn additional algorithms for multiplication or spend that time mastering other content is a decision best made on a case-by-case basis.

When we introduce a new algorithm, research findings show that students benefit if instruction follows the CPA sequence (Gersten et al., 2009). Most current math programs use models to introduce multi-digit multiplication, but few explicitly link these representations to the abstract representations, which is the evidence-based strategy recommended in the IES Practice Guide for students receiving mathematical support (Gersten et al., 2009). Helping students make meaningful connections between visual models and abstract algorithms is the focus of this section.

The Common Core State Standards specify that in fourth grade, students should be able to:

> CC.4.NBT.5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two, two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010).

Multiplication with Rectangular Arrays

The first model mentioned in the standard, the rectangular array, is shown in Figure 9.5. Because they clearly illustrate that multiplication is repeated addition, arrays provide an excellent model for introducing multi-digit multiplication. Students can model the multi-digit problem as an array, following the same process they did when modeling single-digit multiplication facts, and then use repeated addition to find the product.

Figure 9.5 Rectangular Arrays Illustrate Multiplication as Repeated Addition

5 x 12 = 12 + 12 + 12 + 12 + 12 = 60

x x x x x x x x x x x x
x x x x x x x x x x x x
x x x x x x x x x x x x
x x x x x x x x x x x x
x x x x x x x x x x x x

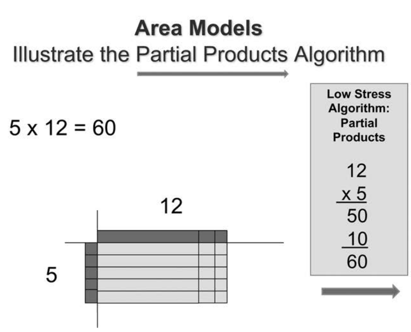
Area Models

The second type of visual representation mentioned in the standard quoted above, the area model, matches the process used in the partial products algorithm, as illustrated in Figure 9.6, and so should be introduced in conjunction with that algorithm.

The partial products algorithm for multiplication is similar to the partial sums algorithm previously described for addition. Students need a solid understanding of place value to successfully execute this algorithm, so place value should be reviewed before introducing the partial products algorithm. To multiply a one-digit number times a two-digit number using the left-to-right approach, as shown in the example, students would first find the total value of the tens, record the partial product, and then find the total value of the ones and record that partial product. Once they have recorded the partial product for each column, they add the subtotals to determine the final product. To reduce confusion, it is helpful to point out the place values of each digit while executing the algorithm.

The partial sums algorithm has two advantages: (1) it builds on students' prior knowledge of place value, and (2) no regrouping is required. Figure 9.6 shows how rate use an

Figure 9.6 Modeling Multiplication with an Area Model



Steps for Multiplying with an Area Model

- ☐ Draw the axis lines.
- ☐ Write the first factor on the left side. Model it.
- ☐ Write the second factor above the top. Model it.
- ☐ Beginning with the largest blocks possible, fill in to form a rectangle.
  - ☐ Multiply *ones* time *tens*. Show the blocks, and record.
  - ☐ Multiply *ones* times *ones*. Show the blocks, and record.
- ☐ Add the partial products. Record the total.
- ☐ Check. Does my answer make sense?

area model to illustrate multiplying a one-digit number by a two-digit number. We recommend giving students a laminated list of steps to follow, and having them check off each step as they complete it. As discussed previously, self-monitoring is an evidence-based practice that has been shown to increase learning outcomes among students who struggle with executive functioning (The IRIS Center, 2020). Students often find it helpful if you begin by modeling each factor with base-ten blocks, so they can clearly see what the dimensions of the rectangle will be. In the example in Figure 9.6, the factors of 5 x 12 are laid out first, so students see what the perimeter of the rectangle will be. Use the largest blocks possible to highlight the place value of each digit. In this example, we use one rod and two unit blocks to represent the factor of 12, and five unit blocks to represent the five. Next, begin to solve the problem by filling in the area of the rectangle, working from left to right. Use the think-aloud process to help students understand the steps. For example, you might say, "This step says to multiply *ones* times *tens*. Here are my ones, and here are my tens, so in this example, I need to multiply five *ones* times one rod, or ten. Five times ten is 50. That's my partial product. I can show it by placing five rods in the rectangle. Now I'll use numbers to record the partial product right here. When I immediately record the partial product, it helps me understand and remember how to solve these problems. OK, now the next step says to multiply *ones* times *ones*. In this example, that means multiplying 5 *ones* times 2 *ones*, which gives me 10 ones. I'm going to use unit blocks to represent this value, rather than a rod, so that it fits into the 5 x 12 rectangle we have outlined. Alright, now I can check that step off. Finally, I will add together all the partial products I wrote down, and record the total."

Note that in the example above, the interventionist continually referenced the steps. Instead of simply saying, "Multiply five times two," the interventionist said, "This step says to multiply *ones* times *tens*. In this example, that means multiplying five ones times one rod, or ten." If we simply tell students what to write, students may become dependent on the teacher to tell them what to do. Our goal is for students to develop a deep understanding of the underlying concepts. Pointing to each step and thinking aloud about what that step means for this problem helps the students master a strategy that they will be able to apply independently when solving any problem of this type.

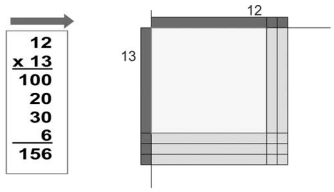
Figure 9.7 shows how area models can be used to multiply a two-digit times a two-digit problem. In the online resources for this chapter, we have provided an additional example of teaching students how to use an area model and the partial products algorithm when multiplying by two-digit numbers.

The Standard Algorithm for Multiplication

By the end of fifth grade, students should be able to "fluently multiply multi-digit whole numbers using the standard algorithm" (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). The standard algorithm for multiplication is the way most adults learned to solve multi-digit multiplication problems. It is worked from right to left, and requires "carrying," or regrouping from ones to tens, tens to hundreds, and so on. Neither arrays nor area models accurately mirror the steps in the standard algorithm. To help students understand the meaning behind the steps in the standard algorithm, we need to use a process similar to that introduced when modeling the standard algorithms for addition and subtraction, where students arrange blocks on place-value mats in the same sequence used in the algorithm.

To introduce the standard algorithm for multiplication, we recommend following the CPA continuum and beginning at the concrete level. The purpose of using concrete representation

Figure 9.7 Modeling Two-Digit Multiplication with an Area Model



Steps for Multiplying with an Area Model

- □ Draw the axis lines.
- □ Write the first factor on the left side. Model it.
- □ Write the second factor above the top. Model it.
- □ Beginning with the largest blocks possible, fill in to form a rectangle.
    - □ Multiply *tens* times *tens*. Show the blocks, and record.
    - □ Multiply *tens* times *ones*. Show the blocks, and record.
    - □ Multiply *ones* times *tens*. Show the blocks, and record.
    - □ Multiply *ones* times *ones*. Show the blocks, and record.
- □ Add the partial products. Record the total.
- □ Check. Does my answer make sense?

is to give meaning to the abstract algorithm, but this goal can be accomplished only if we explicitly connect each step in the algorithm with the concrete manipulation. Therefore, when we model the standard algorithm, we begin in the ones column. Students should first use blocks to model the values in the ones column on a place-value mat, and then record the total on the abstract problem, so they see the relationship between the concrete representation and how they use paper-and-pencil to solve the problem abstractly. After they model and record the product obtained by multiplying ones, they can model the tens and record the result. It is helpful to introduce the standard algorithm using problems that do not require regrouping, so that students systematically master the idea of beginning in the ones column, and then later begin to deal with the process of regrouping. Help students connect regrouping in multiplication with the "making trades" game they played previously, as well as with the process they use for regrouping in addition. See the online resources for a detailed example of the modeling portion of a lesson introducing regrouping using the standard algorithm for multiplication.

In many core materials, students use arrays and area models to illustrate multi-digit problems, but never use manipulatives in a way that matches the steps of the standard algorithm. Without the opportunity to experience this algorithm at the concrete and representational levels, students do not make meaningful connections between the models and the abstract computation. Their lack of understanding is evident in the errors they make. Consider the example below. The correct solution to the problem 13×45 is written first, followed by an example of a common student error.

$$\begin{array}{r} 45 \\ \times 13 \\ \hline 135 \\ 450 \\ \hline 585 \end{array} \qquad \begin{array}{r} 45 \\ \times 13 \\ \hline 135 \\ 45 \\ \hline 180 \end{array}$$

In the second problem, the student omitted the zero that belongs after "45" in the partial product. When students learn to execute the algorithm as a rote procedure, they are often told to add a zero as a "place holder." Students who have a solid foundation in concrete experience know that this zero has meaning; the 1 in the factor 13 represents "1 ten," and we record a zero because we are multiplying 45 by ten, not by just one, so the product is really 450, not just 45. Students are less likely to make this type of mistake if they began at the concrete level, using rods to represent numbers in the tens column and matching their concrete models to each step in the standard algorithm.

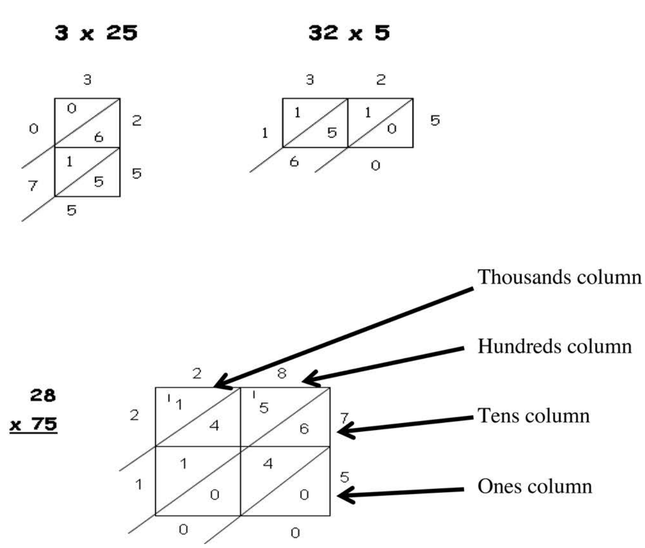
Lattice Multiplication

If students have difficulty keeping the rows and columns aligned when recording their answers in the traditional format, an alternative algorithm called "lattice multiplication" may be useful. With this algorithm, students create boxes in which they record partial products, and these boxes help keep the columns organized. Another advantage of this method is that students do not need to regroup when recording partial products, although they do regroup when combining the partial products to obtain the final product. Figure 9.8 shows

Figure 9.8 Lattice Multiplication

Steps for Using Lattice Multiplication

- Make the boxes.
  - Along the top of your paper, draw a box for each digit in the first factor.
  - If the second factor contains more than one digit, add another row of boxes for each extra digit.
  - Draw diagonal lines in the boxes, extending beyond the boxes on the left side.
- Write the first factor along the top of the boxes, placing one digit above each box.. Write the second factor along the right side of the boxes, placing one digit to the right of each box..
- Multiply the digits in each factor. Record the answer by writing one numeral in each half of the boxes.
- Add the numbers diagonally, regrouping if necessary. Record the answers along
- the bottom & left sides of the boxes.



how to solve a problem using the lattice method. Students write the factors along the top and right edges of the form, record partial products inside the boxes, and write the final product along the left side and below the bottom edge of the boxes. Place-value columns run diagonally, with the ones columns in the lower right corner, and progress across the boxes to the upper left corner. To model lattice multiplication, interventionists can use a format similar to that described for introducing the standard algorithm. To make their place-value mats more accurately match the columns in the lattice, students can tip the mats on a diagonal when modeling lattice multiplication.

Multi-Digit Division

The instructional sequence for introducing multi-digit division parallels the expectations for solving multi-digit multiplication. According to the Common Core State Standards, at the fourth-grade level, students should be able to divide up to four digits by a one-digit divisor, using equations, rectangular arrays, and/or area models to explain their calculations. These are the same models used to represent multiplication, and connecting multi-digit division to students' previous experiences with multiplication can enhance and solidify their conceptual understanding of both operations.

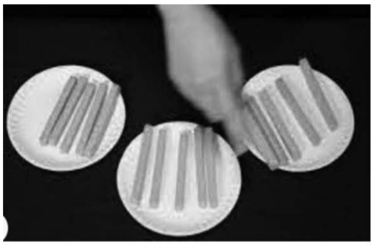
The Standard Algorithm for Division

The Common Core State Standards use arrays and area models to introduce division, but by the end of sixth grade, students are expected to fluently divide multi-digit numbers using the standard algorithm (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). The steps used to develop the rectangular arrays and/or area models that students use in fourth grade do not match the steps used to execute the standard algorithm, and many textbooks designed for the core curriculum do not provide a concrete model that matches the standard algorithm. We recommend following the CPA continuum by providing a concrete model of the standard algorithm for division. The purpose of using concrete representation is to give meaning to the abstract algorithm, but this only happens when the concrete model uses the same steps that are taught in the algorithm.

The "equal groups" model, similar to that shown in Figure 9.9 for modeling partitive division facts, is an excellent way to demonstrate the meaning behind the standard algorithm for dividing larger numbers. By using the same model students used when they first learned to divide, we help connect the new information to students' previous experience dividing single-digit numbers. Just as they did when modeling division of basic facts, students can use paper plates or mats to represent the divisor and base-ten blocks to model the dividend. Starting with the largest blocks, they distribute blocks evenly until no additional equal groups can be created. After students execute a step with the blocks, they should stop and record their work using the standard algorithm format. If they have blocks left over, they can make a trade for the next-sized blocks, evenly distribute all the blocks of that size, and record the results. The process is repeated until all the blocks are evenly distributed. Once again, we recommend giving students a list of steps for the process, and teaching them to follow the steps as they divide. See Figure 9.9.

Let's say the problem shown in Figure 9.9 is to divide 312 baseball cards into two groups. Step one is to model the problem. Students can use base-ten blocks to model 312 with three flats (hundreds blocks), one rod (tens block), and two units. Use two plates to represent the two groups. Once we have modeled the problem, we can check off that

Figure 9.9 Modeling Equal Groups in Division



Modeling Division
164÷ 3

Steps for Solving Division Problems

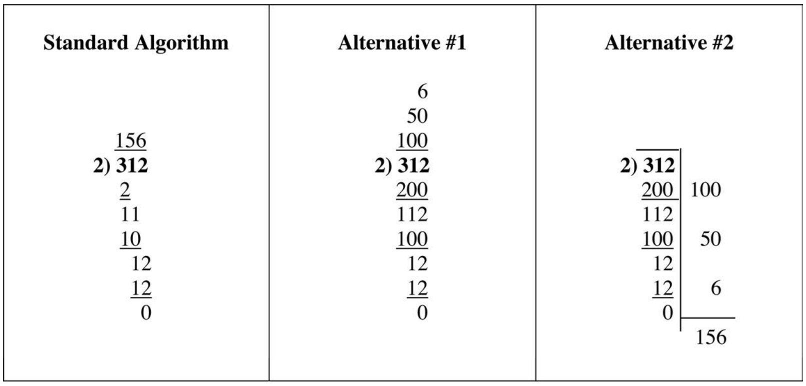
1. Model the problem.
   – How much do I have? Show the dividend (total).
   – How many groups? Show the divisor.
2. Distribute the blocks equally among the groups.
   – Start with the largest blocks.
   – Share equally. Record.
   – Blocks left over? Make a trade.
   – Count <u>all</u> and record.
   – Move to the next largest blocks and repeat.
3. Check your answer

step, and then move to step 2. Step 2 says to begin with the largest blocks, and share them equally. In this problem, the largest blocks were the three flats that represented 300. We place one flat in each group, but then we have to stop because the remaining block cannot be shared equally. Record that each group now has one flat, or 100. We have one block left over, so we can trade that leftover flat for ten rods. If we combine those rods with the rod we had at the beginning, we see that we now have 11 rods. Record this step on the abstract problem. Now we can share those rods equally. We can put five rods in each group, but then we have to stop because the remaining block cannot be shared equally. Record the five rods we put in each group. Trade the leftover rod for ten units. If we combine those units with the original two units, we have 12 units. We can share 12 units equally between our two groups. We put six units in each group, so we record that number on the abstract number problem. That means that when we divide 312 baseball cards into two equal groups, we have 156 cards in each group. We can check our work by counting the blocks in each group, to make sure the numbers on the number problem match what we show with the blocks. In the online materials for this chapter, we provide a detailed example of how this procedure can be used to introduce the standard algorithm for division.

Alternative Algorithms for Division

Students sometimes struggle with the standard algorithm we just described because it forces them to approach division in a piecemeal fashion, recording one small portion of the dividend at a time. For example, in the problem modeled above, the teacher has students divide 312 baseball cards into two groups. In the standard algorithm shown at the left in Figure 9.10, the teacher would model the problem with base-ten blocks by placing one flat in each group, and then record this step by writing 1 in the quotient. Although the digit 1 represents a flat with a value of 100, in the standard algorithm we traditionally only write the first part of the number, and the zeroes are omitted. The next step is to multiply this partial quotient times the divisor. In this example, the divisor is 2, and the teacher would record the product as 2. In reality, she multiplied one flat representing 100 baseball cards times two groups and so has now distributed 200 of the baseball cards. In her explanation, the teacher might clarify the value of each recorded digit, but the algorithm itself uses a shortcut method of recording that is very abstract because it omits the zeroes. To make the process more transparent, some programs teach an alternative recording method that retains the zeroes, with the result that the written record more obviously reflects each

Figure 9.10 Division Algorithms



number's value. Instead of recording the first partial quotient as 1, students using the alternative algorithm would write its entire value of 100, as illustrated in the two versions shown in Figure 9.10. In these algorithms, the partial quotients can be recorded in a pyramid fashion above the problem, as shown in the first alternative example, or to the right of the problem, as shown in the second example. Both versions retain the zeroes, so the numbers students write clearly indicate the actual value. Because these partial quotient algorithms employ a more holistic approach, students may find them easier to understand and remember. The "equal groups" method described for modeling the standard algorithm is equally effective when modeling these alternative algorithms.

Intensifying Instruction During Interventions

Although the process for teaching multiplication and division to students who receive tiered supports is similar to instructional strategies presented in core (Tier 1) instruction, there are important differences. Many educators who provide math interventions do not have access to a validated program where intensive intervention practices are already built into the materials, while others work with students who require even more individualized supports. Ideas for intensifying existing materials to meet the needs of learners receiving tiered support were discussed in the pages above. Here is a summary of some of the many ways to intensify instruction during interventions. Note that these are the same suggestions for intensifying instruction that were presented for intensifying instruction for addition and subtraction.

1. Use systematic instruction. Select objectives carefully. Sequence them from easiest to hardest, and make sure that pre-requisite skills are mastered before introducing more complex content. If students struggle, objectives can be further broken down into component parts or steps. If a student struggles to complete all the steps in a single lesson, then the lesson could be broken down to focus on only one or two steps each day. Although it will take longer to introduce the complete procedure, this approach often saves time in the long run because it reduces the need for reteaching. To avoid overwhelming students' cognitive capacity, pace instruction so that students solidify their understanding of one concept or skill before introducing another.
2. Use explicit instruction. Follow the guidelines described in Chapter 5. If the available materials do not use this high-leverage practice, then modify the lesson to include all the elements of explicit instruction.

3. Give students a written list of steps to follow, and teach them to refer to the list as they work. Many students who struggle with mathematics have deficits in executive functioning. Teaching them to monitor their progress by checking off steps has been shown to increase achievement.
4. Follow the CPA continuum. Always begin at the concrete level, and allow students sufficient time exploring math with manipulatives before expecting them to solve problems using only abstract words and numbers. Explicitly connect the concrete and pictorial representations to the abstract algorithm to build deep understanding. When students can explain the meaning of each step, they are ready for instructors to fade the concrete and visual supports and focus on developing procedural fluency with abstract representation.
5. Use precise academic language when you model mathematical procedures. Emphasize vocabulary in each lesson, and have students practice using the academic vocabulary themselves. Supplementing verbal language with gestures has also been shown to increase understanding and retention for some students.
6. Have students explain what they are doing, and why they are doing it this way. Asking students to explain their reasoning helps them solidify understanding, and also provides valuable formative assessment information that can be used to refine instruction. Core materials increasingly stress the importance of communication in mathematics. Too often, educators who work with struggling learners have encouraged students to use tricks and follow steps by rote, without building deep understanding. Encouraging students to explain their own reasoning, and to understand and critique the reasoning of others, is important to develop mathematical proficiency.

Summary

In this chapter, we provided suggestions for developing students' conceptual understanding through the use of explicit strategies and by systematically linking concrete and visual representations to the abstract algorithms used when multiplying and dividing whole numbers. In the next chapter, we focus on strategies to help students develop computational fluency, because students who are proficient in mathematics not only understand what they are doing, but can also solve problems efficiently.

10
Fact Fluency

Fluency in mathematics means that students can perform computations accurately and effortlessly. However, fluency involves more than speed. Fluent readers can recognize words by sight, and also have strategies for figuring out unfamiliar words, read with expression, and understand what they are reading. An individual who is fluent in a foreign language can speak quickly and smoothly, has a flexible vocabulary and so can select the optimal word or phrase, and easily comprehends what others are saying. Mathematical fluency likewise includes many components. The Common Core State Standards for Mathematics (CCSSM) defines procedural fluency as "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (CCSSI 2010, p. 6). According to the National Council of Teachers of Mathematics (NCTM):

> Computational fluency refers to having efficient and accurate methods for computing. Students exhibit computational fluency when they demonstrate *flexibility* in the computational methods they choose, *understand* and can explain these methods, and produce accurate answers *efficiently*. The computational methods that a student uses should be based on mathematical ideas that the student understands well, including the structure of the base-ten number system, properties of multiplication and division, and number relationships" (NCTM 2000, p. 152).

Fluency with basic facts is a pre-requisite for all other computational fluency. According to Baroody (2011), basic fact fluency is the efficient, accurate retrieval of single-digit calculations. Basic facts include the 100 addition facts formed by combining two single-digit addends, the 100 related subtraction facts, the 100 multiplication facts formed by two single-digit factors, and their related division facts. Because zero cannot be used as a divisor, there are only 90 division facts. Although many math programs include addends or factors of 10, 11, and 12 in their basic fact practice, problems formed with two-digit numbers are not technically basic facts. Figure 10.1 shows the 390 basic facts. When students can solve these 390 facts quickly and effortlessly, they will be able to more easily perform computations with larger numbers.

In addition to negatively impacting students' problem-solving ability, lack of competence with basic facts has been shown to negatively affect students' attitudes toward

Figure 10.1 Basic Facts

100 Basic Addition Facts

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

100 Basic Subtraction Facts

-	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

100 Basic Multiplication Facts

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

90 Basic Division Facts

÷	0	1	2	3	4	5	6	7	8	9
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

mathematics, including decreased self-efficacy and increased anxiety (Miller, 1996; Tucker, Singleton, & Weaver, 2002). Because automaticity is essential for mathematical proficiency, the IES Practice Guide recommends, "Interventions at all grade levels should devote about ten minutes in each session to building fluent retrieval of basic arithmetic facts" (Gersten et al., 2009).

Baroody (2006) suggests that students develop fact fluency in three phases. Phase One involves modeling and/or counting to find the answer. In Phase Two, students derive answers using reasoning strategies based on known facts. For example, if a student knows that $2 + 2 = 4$, then that student can reason that $2 + 3$ is one more than $2 + 2$, so $2 + 3 = 5$. Phase Three is mastery, where a student knows that $2 + 3 = 5$ without having to pause to figure it out. Research findings indicate that students must move through these phases in sequence. Traditional approaches have often jumped from Phase One directly to Phase Three, omitting strategy instruction and instead immediately focusing on memorization. However, spending time developing strategic thinking has been shown to improve fact mastery (National Research Council, [NRC] 2001; Kanive et al., 2014; Woodward, 2006).

In Chapter 8, we discussed methods for developing conceptual understanding of the operations. This represents Phase One of Baroody's three-phase process. Students with a solid conceptual foundation understand that addition involves joining quantities, while subtraction means separating or comparing quantities. Multiplication is repeated addition, while its inverse, division, involves repeated subtraction. Students who score in the proficient range on universal screening measures are able to explain these big ideas and represent operations easily and accurately using a variety of representational forms. Given a number problem, they can represent it using objects, pictures, or words. Given a word problem, they can express it in numbers, act it out, or illustrate it graphically. Before students work on efficient computation, they need a solid understanding of the underlying process, so the procedures described in the previous chapters for modeling basic facts are prerequisites for this chapter.

Students who have a basic understanding of the operations can begin to develop *flexibility.* Flexibility means knowing several different ways to solve a problem, and being able to select the method that is most efficient for that situation. There are many strategies for finding the answer to a math fact problem. We devote the first portion of this chapter to introducing a variety of strategies for computing the basic facts in each operation, and discuss ways to help students select the most efficient strategy for a given problem. This is equivalent to Phase Two of Baroody's three-phase process.

Automaticity is the ability to provide a correct answer without consciously thinking about it. This represents Phase Three of Baroody's progression. Just as a fluent reader recognizes words automatically, without consciously thinking about them or sounding them out, so too can students who have developed automaticity with basic facts instantly state the answer to a basic fact problem. Typically, students are considered fluent when they can identify math facts within two to three seconds (Burns et al., 2010; Stickney et al., 2012). The CCSSM expectations highlight the importance of automaticity by including standards that explicitly require students to know from memory all sums or products of two one-digit numbers. When students master this standard, they no longer have to devote mental energy to solving facts, and can therefore focus their cognitive resources on higher level computational skills. Historically, educators sometimes de-emphasized strategies and rushed to memorization. More recently, there has been a tendency to de-emphasize the mastery phase. However, research suggests that students need both. Combining strategy instruction with practice activities to build automaticity increases achievement outcomes (Morano, Randolph, Markelz et al., 2020; Ok & Bryant, 2016; Woodward, 2006). The final part of this chapter focuses on developing automaticity with basic facts.

Developing Strategies for Solving Basic Facts

Research has not yet established an optimal sequence for teaching basic facts (Hudson & Miller, 2006), but many experts recommend organizing instruction around specific strategies. Counting by fives, using the add-1 rule, using reciprocals, and a variety of other methods that facilitate efficient retrieval of a particular group of facts can reduce cognitive load and facilitate computational fluency (Bley & Thornton, 2001; Purpura et al, 2016; Van de Walle et al., 2019). Executing a strategy consumes short-term memory capacity (Baddeley, 1980; Case, 1985), so interventionists should be judicious in the use of strategies with students who have deficits in short-term memory. When students first encounter a new strategy, using the strategy may consume most, if not all, of their cognitive capacity. Extensive practice may be necessary before students can execute a strategy fluently and automatically (Pressley & Afflerbach, 1995). Once students understand and can successfully execute a particular practice, they will then need carefully planned, massed, and distributed practice in order to use the strategy efficiently without taxing their working memory capacity.

In the next sections, we discuss a variety of strategies for solving basic fact problems. We recommend that interventionists follow the principles of systematic instruction and chunk math facts by strategy, introduce one strategy at a time, and provide plenty of practice time before introducing additional strategies. To compute fluently, students first need to understand how a strategy works, and then practice using the strategy, and finally learn to discriminate when to use the strategy and when a different strategy might be more appropriate.

Addition Facts

Counting All and Counting On

When students first learn to add, they use concrete objects to represent the first addend, and then represent the second addend, and finally they join the two sets and count the total, starting from one. This is referred to as "counting all." Students who struggle with basic facts often continue to rely on counting every object in order to solve basic fact problems long after their peers have committed these facts to memory or developed more efficient strategies to help with fact retrieval (Siegler, 1988). For example, given the problem 6 + 7, a student who struggles with basic facts will continue to hold up six fingers or draw six tally marks, then add seven more, and then join the two groups and count each object one at a time, beginning with the number 1 and continuing until all thirteen objects have been counted. This process produces the correct answer, but it is an inefficient strategy. To become more efficient, students first need to learn to "count on" from a given number, so that when they join two sets, they no longer need to recount everything but instead can begin counting on from the first number to obtain the total. Most students develop this strategy independently as early as age four (Siegler & Jenkins, 1989). However, second-grade students who struggle with mathematics may still not have mastered this skill (Tournaki, 2003). Research suggests that systematic and explicit instruction can help these students learn to use the strategy to facilitate fluent fact retrieval (Gersten et al., 2009).

Several activities are useful for practicing the counting-on strategy. One of these is "round robin" counting, where one student begins counting aloud. After that student counts "1, 2, 3, 4," tell her to stop and ask a second student to continue counting "5, 6, 7, 8," and so on. Another effective activity for practicing the counting-on strategy is to draw a large number line on the floor. Students can stand on a number and then count on as they step down the number line. They can also use the ten-frames that were introduced in Chapter 7 to practice counting-on. Have students represent a number like six on the ten-frame, then add one more counter to change the number to seven, then to eight, and so forth. Initially, students will need to recount all the counters beginning with the first one, but with practice they will be able to count on from the last number shown. To model numbers larger than ten, use one full ten-frame card to model the ten, then begin filling a second card to represent 11, 12, and so on. Sequencing activities can also help students learn to generate the next number without having to recount from one. For example, students can sequence a set of ten-frame cards in order from one to ten and then use the cards to practice counting forward out loud. To provide an added challenge, turn over one card in the sequence and have students identify which number was hidden.

Plus-One and Plus-Two Facts

Once students understand the process of counting on, they can use it to identify facts that are one more than a given number, and then facts that are two more than a given number. Students who can count-on efficiently can solve plus-one and plus-two facts almost instantly, so counting-on can be a very effective strategy for these facts. Counting-on is not a recommended strategy for facts requiring adding three or more, because it takes too long and so can interfere with computational fluency. There are 32 addition facts where one of the addends is one or two, so almost one third of the 100 addition facts can be calculated using this simple strategy. See Figure 10.2.

To introduce plus-one and plus-two facts, begin with an activity similar to the oral counting introduced for practicing counting on. Call on one student to start counting aloud, beginning at 1 and continuing until the teacher says, "Stop." Then select another student, who says the next number in the sequence and then states the complete number fact. For example, the first student might count "1, 2, 3, 4." Then the teacher says "Stop" and points to another student. The second student says, "Five. Four plus one equals five." When practicing plus-two facts, the second student would say the next two numbers in the sequence and then state the complete number fact: "Five, six. Four plus two equals six."

A number line drawn on the floor is also a useful tool to practice +1 and +2 facts. Let a student stand on a number and then tell him to add one more or add two more. The student steps on the answer and states the complete math sentence out loud. For example, to model 4 + 1 the student would stand on 4. When the teacher says, "Add one more," the student would step or jump to 5 and say, "Four plus one equals five."

Ten-frame cards can also be used to practice plus-one and plus-two facts. Have the students represent a one-digit number and then challenge them to calculate what the answer would be if they add one more to that number. Write the fact problem on the board so they connect the concrete experience with the ten-frames to the abstract number problem. For example, write 4 + 1 on the board, and let students represent the 4 on their ten-frame mat. Challenge them to predict what the answer will be when they add one more and then let them add the additional counter and count to confirm their prediction. Once students can use counters to accurately predict the results of adding one more to their ten-frame boards,

Figure 10.2 +1 and +2 Facts

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

try flashing a ten-frame card and asking them to state one more than the amount shown. The same activity can then be repeated at the abstract level by replacing the ten-frame cards with numeral cards. A calculator can also be used to practice +1 and +2 facts. Let students enter a fact problem such as 6 + 1, try to predict the result, and then press the equals sign to check their prediction. Once students can quickly and accurately identify the sum when a number is increased by one, begin working on adding two more, using the same strategies. To solidify these facts in long-term memory, students will need additional practice activities. Flash cards and worksheets can be used for practice, but most students will be more motivated by games that allow them to practice their facts with classmates. In the online materials, we provide ideas for using games to develop computational fluency. The first two activities in that section describe board games and egg carton games to practice the "+1" and "+2" facts.

In the core curriculum, plus-one and plus-two strategies are typically introduced simultaneously. One way to intensify instruction is to introduce content in smaller chunks, so interventionists may find their students benefit if they introduce plus-one facts first, and wait to work on plus-two facts until after students are proficient with plus-one facts. Core materials may also move quickly into problems where students are solving for missing middle addends, instead of solving for the sum. For example, instead of presenting the problem as "3 + 1 = ?," the problem may be written as "3 + ? = 4" or "? + 1 = 4." Students must eventually master problems that use all these formats, but introducing each new format separately, with opportunity to practice and consolidate what students have learned before changing the format, is another way to intensify instruction for those individuals who require tiered support.

The Commutative Property of Addition

The commutative property of addition states that $a + b = b + a$. Therefore, if 7 + 5 is known, then 5 + 7 is also known. Instead of memorizing 100 addition facts, students who understand this property can learn just 50 facts and automatically solve the other 50. Students should first experience the commutative property at the concrete level by creating two sets of objects and comparing the results when they add the facts together, first beginning with one addend, then beginning with the other addend. Repeated experiences using concrete objects and pictures will help them recognize that the answer will be the same no matter which order they use to solve the problem. Dominoes provide an excellent visual representation of the commutative property, because a domino can be flipped backward and forward without changing the total quantity. Younger students will enjoy practicing the commutative property with the "Fishy Facts" activity described in the online materials. Having students explain the commutative property in words will provide additional reinforcement and help consolidate understanding.

Facts with Zero

There are 19 facts that have zero as one of the addends (see Figure 10.3). Zero is the identity element in addition, because if you add zero to any number, the result is your original number (a + 0 = a, or 0 + a = a). Students sometimes find the idea of adding zero confusing, and they will benefit if the concept is illustrated in multiple ways, including objects, drawings, and word problems. Students who have been introduced to both addition and multiplication sometimes confuse the effect of using zero in these operations. Adding zero to a number has no effect on the original quantity, but multiplying by zero results in a product of zero. Again, providing concrete and visual representation can clarify the difference in the

Figure 10.3 Zero Facts

The Identity Property of Addition: a + 0 = a

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

two operations. The games suggested to practice plus-one and plus-two facts can be easily adapted to practice zero facts.

Ten-Sums

When given a group of numbers to add, experienced mathematicians often try to find combinations that total ten. Although there are only nine combinations whose sums total ten (see Figure 10.4), they are so useful that it is worth spending time helping students master these facts. Since nine facts is a greater quantity than any student is likely to be able to hold in working memory, interventionists are encouraged to intensify instruction by introducing one or two of the ten-sums and giving students time to explore them, and then systematically introducing additional ten-sum facts.

Ten-frames provide a valuable visual representation to support acquisition of ten-sum facts, because when a number less than ten is represented on the frame, the empty spaces illustrate the number of items needed to make ten. Students can practice representing a number on the ten-frame and then deciding how many more counters they must add to reach ten. As students gain proficiency, you can flash a ten-frame card or a written numeral card and ask them to decide how many more would be needed to make ten. Number lines provide another valuable tool to help students master ten-sums. The MathLine described in Chapter 7 uses a red ring to highlight ten and all multiples of ten, so if students represent the first addend on MathLine, they can easily see how many more are needed to make ten. Developing a mental number line appears to be a critical component of numerical reasoning (Tarver & Jung, 1995). For additional practice, students can play the games described in the online materials, including Finding Ten-Sums, Ten-Sums Fish, Guess My Hand, and Toss 'n' Cross.

Figure 10.4 Ten-Sums

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 10.5 Near Tens

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Near-Tens

The 18 facts referred to as near-tens include all those facts whose sums are one more or one less than ten, as shown in Figure 10.5. For example, when presented with the fact 7 + 4, a student who knows that 7 + 3 = 10 can determine that, since four is one more than three, the sum will be one more than ten. Similarly, since two is one less than three, 7 + 2 must result in an answer one less than ten. Again, ten-frames and MathLine provide concrete and visual representation to support the students' acquisition of these facts. Students must use their knowledge of ten-sums to calculate the near-tens, so this strategy is best introduced after students have already mastered ten-sums. Core materials often introduce near tens after they introduce ten-sums. While this represents a logical sequence, students need many opportunities to practice ten-sums before they become fluent with those nine facts. Systematically pacing instruction so students have enough time to master ten-sums before introducing near-tens is an effective way to intensify instruction.

Doubles

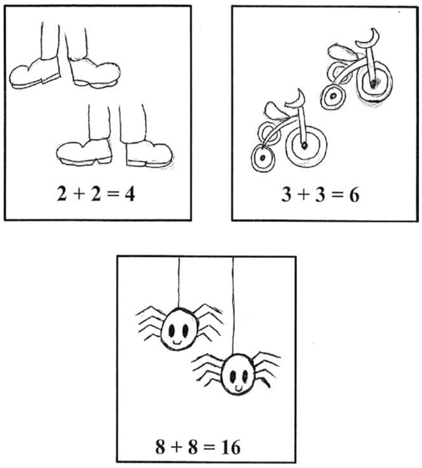
There are ten facts that are formed by doubling single-digit numbers, as shown in Figure 10.6. To help them remember these facts, students can create drawings of real-life examples of doubles, as shown in Figure 10.7. Including the written number fact with the drawing will connect the visual and abstract representational forms and further support computational fluency.

Some authors suggest using doubles pictures that show the entire fact in a single object. For example, insects have three legs on one side of the body and three legs on the other

Figure 10.6 Doubles

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 10.7 Modeling Doubles



side, so one insect can be used to illustrate 3 + 3 = 6. We can also use the entire insect to illustrate the number six. If we do that, then two insects show that 6 + 6 = 12. While either format works for addition facts, if we choose the second option, then our doubles drawings also apply later when we introduce multiplication facts. If students associate an insect with the number six, then three insects have 18 legs, four insects have 24 legs, and so on. Using consistent pictures reduces the memory load and facilitates generalization, which can help students master the multiplication facts more quickly.

Calculators can also be used to practice doubles. If you first enter the "double maker" (2×=), then a student can enter a one-digit number like 6, predict the answer of the doubles fact (6 + 6 = 12), and then press the equals sign to check the prediction. Because doubles are often awarded special significance in board and dice games, students who have experience with these games may more easily memorize facts that involve doubles. In the online materials, we describe two games that students can play to practice their doubles facts—Egg Carton Doubles and Double Trouble.

Near-Doubles

The 18 facts referred to as near-doubles include all those combinations where one addend is one more than the other addend, as shown in Figure 10.8. To calculate the sum of near-doubles, students double the smaller digit and then add one more. This strategy therefore requires students to already know their doubles facts and also to have mastered

Figure 10.8 Near Doubles

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

the plus-one strategy. It can be introduced by presenting a list of near-double facts and then engaging the students in a discussion about how they can use facts they already know to solve these new facts. Students can practice computing near-doubles by rolling a single die and stating the near-double fact that can be created using the number rolled. For example, if a student rolled a three, he would say, "Three plus four equals seven." The games described to practice doubles can be adapted to practice near-doubles by using this procedure. For example, to use the Double Trouble game described in the online materials to practice near-doubles, students roll a pair of dice as explained in the directions; when they roll a double, they transform it into a near-double and write the near-double fact on their paper. The student who records the most near-doubles in five minutes wins the round.

Facts Solved by Making a Ten

If students can decompose two-digit numbers into tens and ones, they can use this knowledge to solve fact problems involving larger numbers like eight or nine. There are 20 addition facts where one addend is eight or nine. See Figure 10.9. To solve these problems, students begin with the eight or nine and count up to ten, and then add on the remaining amount to obtain the total. For example, to add 9 + 6, count up one from nine to ten. Instead of 9 + 6, we now have 10 + 5, and students who understand the base-ten number system will recognize that 10 + 5 is another way of saying 15. An example of this strategy is provided in Common Core Standard 1.OA.6, which states that students should be able to use decomposition strategies to solve a problem like the following: $8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14$. Manipulating discs on ten-frames provides a physical model that can help students understand this concept. Students can also model the process on a number line or on MathLine, just as they did earlier when finding ten-sums. The Make-a-Ten War game described in the online materials describes a game that uses this strategy to practice solving facts containing addends of eight or nine. Since students must use their knowledge of ten-sums in making tens, this is another strategy that is best introduced after students have already mastered ten-sums.

The Leftovers

After students have mastered the addition fact strategies described above, there are four facts remaining. If students apply the commutative property to these facts, then they really only have to learn two more facts to have mastered all 100 basic addition facts. See Figure 10.10.

Figure 10.9 Facts Solved by Making-a-Ten

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 10.10 The Leftovers

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Strategies for Subtraction Facts

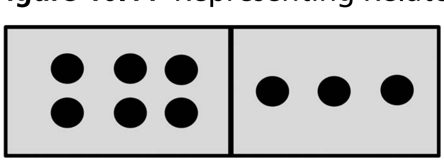
Subtraction is the inverse of addition. There are 100 addition facts formed by combining two one-digit numbers, and 100 related subtraction facts formed by reversing the process, as shown in Figure 10.1. To model subtraction, we generally teach students to represent the total amount, cross off the amount to be taken away, and then count to determine how many are left. This provides an accurate representation of the subtraction process, but it is an inefficient strategy. Students who continue to rely on such counting strategies will struggle when faced with more advanced mathematics. Fluent computation requires that, once students understand subtraction strategies, they eventually memorize the subtraction facts or develop an efficient strategy for solving the problem.

Related Facts

If students have mastered an addition fact, we can use the inverse relationship between addition and subtraction to help them solve the related subtraction fact problem fluently. For example, students who know that $4 + 3 = 7$ can use this knowledge to determine that $7 - 3 = 4$ and $7 - 4 = 3$. Dominoes provide a great visual illustration of fact families and can be used to help students connect subtraction to known addition facts. Show students a domino and have them state the addition and subtraction facts represented on the domino, as illustrated in Figure 10.11.

For example, the domino shown in Figure 10.11 illustrates $6 + 3 = 9$. After students identify the addition fact, cover the dots on one side of the domino and discuss the resulting subtraction fact. If we cover the six dots on the left side of the domino, we have illustrated the subtraction fact $9 - 6 = 3$. Once students are comfortable with this process, try showing them just half a domino while keeping the dots on the other side hidden. Tell them the number of dots on the complete domino, and see if they can determine the number of

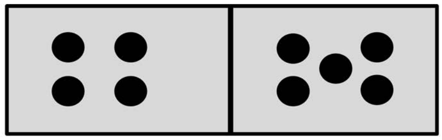
Figure 10.11 Representing Related Facts



This domino can be used to illustrate the following facts:

6 + 3 = 9	**9 – 3 = 6**
3 + 6 = 9	**9 – 6 = 3**

Figure 10.12 Practicing Related Facts



Use the domino to solve these fact problems:

4 + 5 = □	9 – 5 = □	5 + □ = 9
4 + □ = 9	9 - □ = 4	4 + 5 = □
□ + 5 = 9	□ + 4 = 9	9 – 5 = □
5 + □ = 9	□ - 5 =4	5 + 4 = □
□ + 4 = 0	9 – 4 = □	□ - 5 = 4
9 – 4 = □	4 + □ = 9	9 - □ = 4

dots that are hidden from view. For example, if the complete domino contains two dots on one side and six on the other, hide the two dots, show them six dots, and tell them there are eight dots in all. Challenge them to identify the missing addend and then to state the subtraction fact you have illustrated: $8 - 2 = 6$. A worksheet can be created that uses the domino pattern to provide focused fact drill. Create an entire page of problems that all revolve around a single domino, and let students practice associating the two numbers on the domino with the four facts that can be formed using that domino. For example, if one side of the domino contains four dots and the other side contains nine dots, you can use the domino to create the combinations $4 + 5 = 9$, $5 + 4 = 9$, $9 - 5 = 4$, and $9 - 4 = 5$. Create about 20 questions that use these four numbers, with the unknown quantity in different positions, such as $4 + 5 = ?$, $4 + ? = 9$, $9 - 5 = ?$, $9 - ? = 4$, and so on. This process is illustrated in Figure 10.12.

Another activity to help students associate subtraction facts with their related addition facts involves giving each student small addition flash cards. Write a subtraction fact problem on the board, and ask students to hold up the two addition facts that can help them solve the subtraction problem. For example, if you write the problem $8 - 3$ on the board, students can hold up the flash cards containing $3 + 5$ and $5 + 3$. This activity is most effective if students only practice a limited number of facts at one time. Additional games to practice related subtraction facts are described in the online materials.

Counting Down: −1 and −2 Facts

Counting down is another way to calculate subtraction remainders. As we have already discussed, counting up is an inefficient addition strategy to use with large addends, and counting down is equally inefficient. The exception is the facts formed by subtracting one or two. The same strategies used to teach students to count up when adding one or two are equally effective when teaching students to count down when subtracting one or two.

There are 32 subtraction facts that can be solved by counting down one or two, so almost one-third of the 100 subtraction facts can be mastered using this strategy. In order to use the counting-down strategy, students need to be able to count backward from ten to solve first-grade facts and from eighteen to solve second-grade facts. The same strategies used to teach counting on can be used to practice counting down. Let one student begin counting backward; then have that student stop and ask a different child to continue. Once students can count backward easily, model how to use this process to solve facts that involve subtracting one. For example, to model 8 – 1, have the student begin at eight, count back one number to seven, then state the entire fact: 8 – 1 = 7. The same process can be used to practice –2 facts. Let students use their bodies to model this process by walking backward on a large number line taped to the floor. The games described in the online materials to practice +1 and +2 facts can be adapted to practice the counting-down strategy.

Subtracting Zero

There are 19 facts with zero as one of the addends (see Figure 10.3). Just as adding zero to a number does not change the total, when we subtract zero from a number the result is the original number (a – 0 = a). Concrete and visual examples will help students develop this concept. The games used to practice –1 and –2 facts can be easily adapted to practice subtracting zero.

Subtracting the Same Number/Subtracting All

When a number is subtracted from itself, the result is zero—for example, 8 – 8 = 0. Core materials sometimes chunk this skill with subtracting zero. Students who require tiered support benefit from systematic instruction, so may master the concept more quickly if it is introduced separately.

Decomposition Strategies

Decomposition strategies involve decomposing a number to create a simpler or familiar fact and then using that fact to solve the harder fact. For example, to find 14 – 5, students can decompose 5 into 4 + 1. They subtract the 4 from 14 to get to 10, then subtract the remaining 1 to obtain the answer of 9. To subtract 15 – 9, students might first subtract 5 from both numbers and then be able to solve the simpler fact that remains: 10 – 4 = 6. Or students could count up from 9 to get to 10 and then realize that they need to count up 5 more to reach 15, so in all they count up 1 and 5 more, which means they count up 6 in all: 15 – 9 = 6. Students who are competent in math frequently employ decomposition strategies when adding and subtracting, but students with a history of mathematical difficulty may struggle with this approach, because it requires holding multiple pieces of information in working memory. All of the subtraction facts can be solved using related addition facts, so students who find decomposition strategies frustrating can obtain computational fluency if they focus on mastering addition facts and then use this knowledge to solve the related subtraction facts.

Strategies for Multiplication Facts

Multiplication involves repeated addition, so helping students connect multiplication to their existing knowledge of addition will facilitate their acquisition and mastery of the 100 multiplication facts. Before focusing on computational fluency, students first need to develop conceptual understanding of the multiplication process, which is the first of Baroody's three-phase process. In Chapter 9, we discussed ways to use counters, number lines, arrays, and area models to create concrete and visual representations of multiplication problems. When students label a representation with both the addition fact and the matching multiplication fact, it helps them connect the two operations.

Traditional methods for introducing multiplication facts have often progressed sequentially through the multiplication tables, beginning with the ×1 facts, and then introducing ×2, and so on. However, beginning with easiest facts and then clustering instruction around multiplication strategies has been found to be more effective (Van de Walle, Karp, and Bay-Williams, 2018). Kling and Bay-Williams (2015) suggest introducing multiplication strategies in the following sequence: first the facts students have already encountered during skip counting, which includes 2s, 5s, and 10s. Next, they recommend introducing ×0, ×1, and squares (e.g., 3 × 3 and 4 × 4). After that, they suggest teaching students to use known facts solve near facts by adding or subtracting a group, halving and doubling, using a square product to solve a near fact, and decomposing a factor. They provide excellent ideas for introducing each strategy, and suggest games for practicing the strategies. Because students who require tiered support benefit from systematic instruction, we have broken these strategies into smaller segments and suggest introducing each one separately. The first three strategies we discuss build on students' previous experience. They can be used to solve more than half of the 100 multiplication facts.

Multiplying ×2

Multiplying by two is often the easiest table for students to understand, so it is the first table we suggest introducing. There are 20 multiplication facts that have two as a factor, as shown in Figure 10.13. These are equivalent to the ten doubles addition facts students should already have mastered, so teaching students to solve ×2 facts involves helping them connect these multiplication facts to their existing knowledge of addition. The same illustrations that students created to illustrate the addition doubles shown in Figure 10.7 can also show the multiplication doubles. Calculators were discussed as a strategy for practicing addition doubles; the same strategy can be applied to multiplication problems that have two as a factor. Press 2 × = to generate multiplication doubles. Skip counting on a number line or hundreds chart is another way to practice ×2 facts. The games Egg Carton Doubles and Double Trouble described for addition in the online materials can also be adapted to help students master the 20 multiplication doubles.

Figure 10.13 ×2 Facts

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Figure 10.14 ×5 Facts

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Multiplying ×5

Twenty multiplication facts have five as a factor, as shown in Figure 10.14. Students who can count by fives can use their knowledge to rapidly calculate products of the ×5 facts. The Tally Up! game described in the online resources can help activate students' prior knowledge of skip-counting by fives. When students are proficient at skip-counting, we can relate multiplying by fives to skip-counting. One easy strategy is to give students a ×5 fact problem and have them hold up the number of fingers indicated by the factor that is not a five, then count the fingers by fives. For example, to calculate 4 × 5 or 5 × 4, students hold up four fingers and then count those extended fingers by fives: "5, 10, 15, 20." Students can also practice skip counting on a number line or hundreds chart.

We can develop real-life connections for the fives table by using the multiplication facts to find the value of a group of nickels. The Counting Nickels game described in online can be used to practice this skill. Counting by fives is also used to tell time, and developing this connection is another way to help students see the real-life applications of the ×5 table. Draw a large clock face with a minute hand and discuss how we count by fives when reading the minute hand. For example, when the minute hand points to 3, it is 15 minutes past the hour. Relate this idea to the ×5 multiplication facts. Show students a flash card containing a ×5 fact, point to the number on the clock face that matches the second factor on the fact card, and state the complete multiplication fact. For example, show the fact 4 × 5, point to the 4 on the clock, and say 4 × 5 = 20. It is 20 minutes past the hour. The Star Points game described in the online materials provides another way to provide meaningful practice of ×5 facts.

Multiplying ×10

The number ten is a two-digit number, and so the tens table does not technically belong in the multiplication facts. However, students are expected to skip-count by tens before they are expected to master multiplication facts, so students who can skip-count by tens can apply this knowledge to multiplication. The strategies suggested for practicing ×2 and ×5 facts can also apply here.

Multiplying ×0 and ×1

There are 36 facts that contain zero or one as a factor, as shown in Figure 10.15. The rules for solving these facts are best developed through concrete and visual representations. Since one is the identity element in multiplication, any number multiplied by one results in a

Figure 10.15 ×0 and ×1 Facts

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

product that is the same as the original number (e.g. 8 × 1 = 8 or 1 × 8 = 8). After students represent the facts in the ×1 table, ask them to identify the pattern and generate the rule for solving these facts. Once they master the x1 facts, you can use a similar process to help them understand the effects of multiplying by zero. If we ask them to illustrate the ×0 table, they will quickly conclude that any number multiplied by zero is zero. Carefully pace this instruction. If we introduce ×0 facts too quickly, before they have solidified their understanding of the ×1 facts, some students will confuse the two concepts. Using concrete and pictorial representation to generate the rules creates deeper understanding than simply telling them the rule, thus helping students apply the rules meaningfully in problem-solving situations. Although the basic concept of multiplying by zero or one seems relatively easy, students often struggle with these facts because they confuse the results of multiplying by zero and one with the effects of adding zero or one. Adding a zero leaves the original number unchanged, while multiplying by zero results in a product of zero. Adding one increases the original number by one, but multiplying by a factor of one leaves the original number unchanged. Again, using concrete and pictorial representation can minimize confusion.

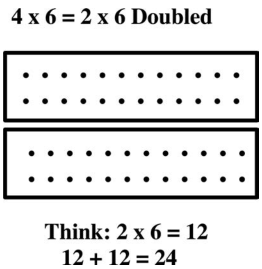
The Commutative Property of Multiplication

The commutative property of multiplication states that $a \ x \ b = b \ x \ a$. Therefore, if 5 × 7 is known, then 7 × 5 is also known. Instead of memorizing 100 multiplication facts, students who understand this property can learn just 50 facts and automatically solve the other 50. Students should first experience the commutative property at the concrete level by modeling the multiplication fact beginning with one factor, and then flipping the fact to begin with the other factor. Repeated experiences using concrete objects and pictures will help them recognize that the answer will be the same no matter which order they use to solve the problem. Arrays and area models provide an excellent visual representation of the commutative property, because both these models can be flipped sideways without changing the total quantity. Having students explain the commutative property in words will provide additional reinforcement and help consolidate understanding.

Multiplication Squares (2 × 2, 3 × 3, etc.)

Squared numbers are easy to illustrate with arrays or area models, and students may find them easier to remember than some of the other facts. These facts can serve as anchors to help them solve other, more challenging facts.

Figure 10.16 Doubling Products



Adding or Subtracting a Group

Once students are fluent with some of the above facts, they can use that knowledge to solve unknown facts. Facts in the ×3 table can be solved by relating them to familiar ×2 facts and then adding one more set. For example, to solve 3 × 4, think 2 × 4 = 8, and then add one more set of 4, to make a total of 12. To solve 3 × 7, think 2 × 7 = 14, and then add one more set of seven to make a total of 21. To find the product of 9 × 6 or 6 × 9, think six sets of ten, and then remove the tenth block from each set of ten. Students think, “Six times ten equals 60, minus six leaves 54.” If a student knows the multiplication squares, then they can use that knowledge to find near squares. For example, to find 5 × 4, the student might say, “I know that 4 × 4 = 16, so 5 × 4 must be 4 more. 5 × 4 = 20.”

Doubling Products

Students who have learned some of the facts in the lower tables can use than knowledge to solve problems with larger factors. For example, facts in the ×4 table can be solved by first finding the related ×2 fact and then doubling the product. For example, 4 × 6 is the same as (2 × 6) + (2 × 6), or 2(2 × 6). See Figure 10.16. Using this strategy requires students to be able to double large numbers like 12 + 12 or 16 + 16. Some students may find this strategy useful, while others may find the mental doubling more challenging.

Multiplying ×9

Students who can skip count by ten can use this knowledge to solve x9 facts. For example, if they know that 6 x 10 = 60, then 6 x 9 must be 6 less, or 54. See Figure 10.17.

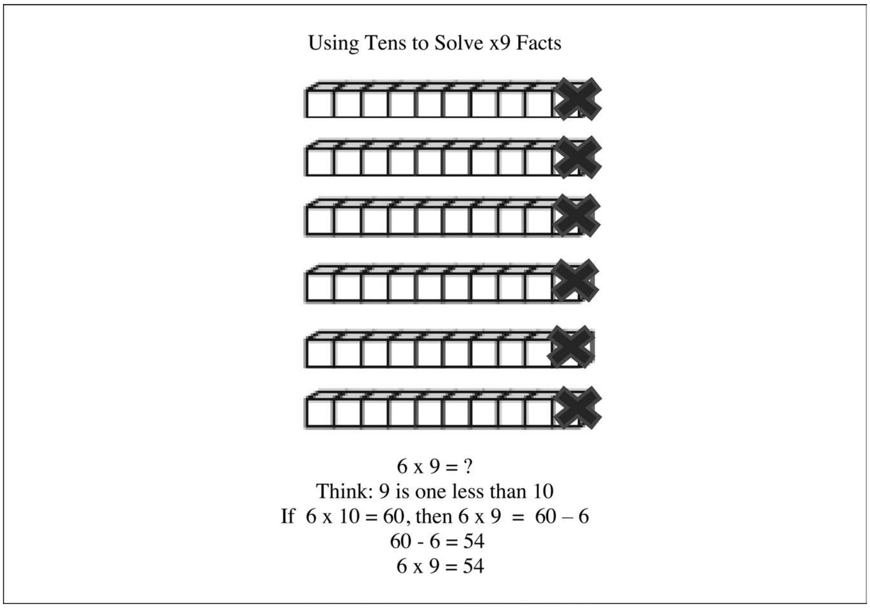
Decomposing a Factor

This strategy involves decomposing one factor to create two known facts, and then combining the products. For example, if the student does not know the solution to 8 × 7, he might decompose the 8 into 5 + 3, because he knows 5 × 7 and 3 × 7. He would then add 35 plus 21 to find the answer of 56. This strategy is often recommended by math educators. However, successful execution of the strategy requires a student to be able to hold multiple numbers in working memory. Since many of the students who struggle with mathematics have deficits in working memory, this strategy may overwhelm their cognitive capacity. Interventionists should monitor carefully to see whether introducing this strategy is helpful or overly taxing.

Strategies for Division Facts

There are 90 division facts students need to master. Because division is the inverse of multiplication, these facts are best learned by linking them to their related multiplication facts

Figure 10.17 x9 Facts

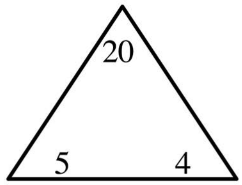


(Fuchs et al., 2005; Fuchs, Fuchs, Hamlett et al., 2006; Fuchs, Seethaler et al., 2008). For example, if students know that $3 \times 4 = 12$ or $4 \times 3 = 12$, they can use this knowledge to determine that $12 \div 4 = 3$ and $12 \div 3 = 4$. Mastering the multiplication facts and then establishing connections between these multiplication facts and their inverse division facts are key to mastering the 90 division facts. The same activities described to help students associate subtraction facts with their related addition facts can be used to link division facts to previously mastered multiplication facts. Figure 10.18 shows an example of the type of worksheet that can help students associate the numbers in related fact families. A game to practice division facts is included in the online materials.

Selecting an Effective Strategy

Extensive research has been conducted on the use of strategies in a variety of domains, and researchers have concluded that students must first learn to execute a strategy and then practice identifying situations in which the strategy could be appropriately applied (Pressley & Woloshyn, 1995). Once students know both *how* to use the strategy and *when* to use the strategy, they still need continued cues and prompts reminding them to use the strategy before its use becomes habitual. Without this scaffolded support, individuals generally return to old, familiar habits, even if these old methods were less efficient. In the case of basic fact computation, students are likely to revert to using their fingers to determine sums of unfamiliar facts. Therefore, interventionists will need to provide systematic practice in strategy selection and application. Each of the activities described in this section focused on a specific strategy, and all of the examples were selected to practice just that strategy. To increase the probability that students will select and use these strategies appropriately in the future, interventionists will need to provide lessons that require students to select a strategy. After students have learned two different strategies, provide them with a set of math facts and ask them to describe which strategy they think would be most appropriate to help them solve a particular fact. For example, after students have worked on facts that use the counting-on strategy (+1 and +2 facts) and the doubles strategy, give them both types of facts and ask

Figure 10.18 Related Facts



Use the numbers in the triangle to solve these related fact problems:

4 x 5 = ☐	20 ÷ 5 = ☐	5 x ☐ = 20
4 x ☐ = 20	20 ÷ ☐ = 4	4 x 5 = ☐
☐ x 5 = 20	☐ x 4 = 20	20 ÷ 5 = ☐
5 x ☐ = 20	☐ ÷ 5 = 4	5 x 4 = ☐
☐ x 4 = 20	20 ÷ 4 = ☐	☐ ÷ 5 = 4
20 ÷ 4 = ☐	4 x ☐ = 20	20 ÷ ☐ = 4

them to select a strategy that would be a good choice for a given fact, and then to explain their choices. To practice this decision-making skill, they can sort facts into piles of facts that would all be solved using the same strategy. They can also be given worksheets containing a variety of facts that would best be solved using two or more different strategies. Instead of asking students to solve the problems, have them identify the most appropriate strategy with each fact and then justify their choices. Even after students can select an appropriate strategy and execute it efficiently, they are still not likely to use it independently when the need arises. Students need prompts to use the strategy throughout the day when situations arise in which a particular strategy would be useful. They will need many prompts and reminders before they begin to generalize a strategy and use it autonomously when computing basic facts.

Automaticity

Phase Three of Baroody's three-phase process is mastery, where students know the answer to a fact problem without having to figure it out. NCTM's emphasis on mastering basic facts highlights the important role of computational fluency in developing mathematical proficiency. Students who know the basic facts automatically are able to focus their attention on problem solving and higher-level computational procedures. In contrast, students who do not know the basic facts from memory must focus their attention on computation and so have less cognitive capacity available for more complex tasks. As Van de Walle explains,

> Fluency with basic facts allows for ease of computation, especially mental computations, and, therefore, aids in the ability to reason numerically in every number-related area. Although calculators and tedious counting are available for students who do not

> have command of the facts, reliance on these methods for simple number combinations is a serious handicap to mathematical growth. (2004, p. 156)

The CCSSM also emphasize automaticity with basic facts, stating that by the end of second grade, students should "know from memory all sums of 2 one-digit numbers," and by the end of third grade, they should "know from memory all products of 2 one-digit numbers" (National Governors Association, 2010).

Students who are successful in mathematics often master math facts through the games and activities used during strategy development, but the majority of students who struggle with mathematics do not master the facts through core instruction. Researchers have determined that students with mathematics disabilities frequently struggle with automaticity (Geary, 2004, 2013). Studies have shown that, by age 12, the average student with a learning disability can recall only one-third as many facts as non-disabled peers (Hasselbring et al., 1988). Although they could compute accurately, the individuals with learning disabilities still relied on counting fingers or tally marks rather than responding to a fact problem automatically. Further research has extended these findings beyond students with disabilities to all students who struggle with mathematics. Individuals who fail to demonstrate mathematical proficiency and who will therefore require tiered interventions typically lack automaticity with basic facts. These students consistently demonstrate extremely slow fact retrieval (Geary, 2004; Geary et al., 2007).

Much has been written about the overuse of "drill and kill" techniques, and teachers sometimes hesitate to spend much time drilling facts. However, drill does have an important role in developing computational fluency. As Van de Walle explains,

> Drill—repetitive non-problem-based activity—is appropriate for children who have a strategy that they understand, like and know how to use but have not yet become facile with it. Drill with an in-place strategy focuses students' attention on that strategy and helps to make it more automatic. Drill plays a significant role in fact mastery, and the use of old-fashioned methods such as flash cards and fact games can be effective if used wisely (Van de Walle, 2004, p. 158).

To master basic facts, students need to focus on just a few facts at a time. Many of the materials and activities that are intended to develop computational fluency actually present too many unfamiliar facts simultaneously to foster fluent retrieval. Think back to the discussion of working memory at the beginning of Chapter 5. Research has shown that the average five-year-old can recall about two items. A typically developing seven-year-old can retain about three items, and a typical nine-year-old can retain about four items. By age 11, retention increases to about five items, and by age 13, the average individual can recall about six items (Pascual-Leone, 1970). These numbers describe the working memory capacity of average learners. We know that individuals who struggle with mathematics frequently have less working memory capacity than their normally achieving peers (Allsopp et al, 2010; Mabbott & Bisanz, 2008; Mazzocco, 2007; Swanson, Jerman, & Zheng, 2009). In practical terms, this means that an individual with a deficit in working memory may hold at least two fewer items in working memory than their typically achieving peers. In other words, a seven-year-old with deficits in working memory can be expected to retain a single item. By age nine, that same individual might be able to hold two items in working memory, and by age 11, the number would increase to three. The implications for mathematical proficiency are devastating. Think about the many steps involved in performing any of the algorithms. A student who must stop to figure out basic facts has no capacity left to remember the steps of the procedure. It is imperative that these individuals can solve basic fact problems automatically, because they need all their attention available to focus on other aspects of the problem, including algorithms, formulas, information presented in word problems, etc. If we apply this brain research to developing computational fluency, addition and subtraction fact practice presented in the core curriculum (Tier 1) should focus on no more than three facts at a time, because that is the cognitive capacity of average second-grade students. Practice activities for multiplication and division facts should focus on no more than four facts at a time, because that is the cognitive capacity of the average fourth-graders who are learning these facts. Since individuals who struggle with mathematics often have less working memory capacity than their normally achieving peers (Allsopp et al., 2010; Mabbott & Bisanz, 2008; Swanson, Jerman & Zheng, 2009), the IES Practice Guide recommends that interventionists working with these students focus on only one or two unfamiliar facts at a time. The two unknown facts can be interspersed with review of known facts, so that a student might practice five or even ten facts at a time, but only two of these should be facts that the student cannot yet compute fluently (Gersten et al., 2009).

To arrange effective practice, the interventionist should first assess each student's fact proficiency and then present instructional activities in such a way that the individual can focus on just two new facts in one instructional session. Flash cards can be used to assess the student's mastery of required facts. As the student answers each fact, the card is placed in one of two piles: (1) facts the student can answer in less than three seconds, and (2) facts the student cannot answer or needs more than three seconds to compute. When creating practice activities for a student, interventionists should include one or two of the unknown facts and then add some of the known facts to fill out the activity and provide ongoing review. Once the student can consistently solve a fact from the unknown pile in less than three seconds, the instructor can add a new fact to practice activities and continue in this manner until the student has mastered all facts for that operation. While this process may seem slow and tedious, students achieve automaticity far more quickly when they experience such focused practice opportunities.

The procedure described above for selecting unknown facts assumes that each individual fact counts as one item in short-term memory, so that students with a capacity of two items should work on just two facts at a time. However, when information is clustered meaningfully, multiple facts may be grouped together and still count as just one item in working memory. Consider the analogy of a small change purse that is only large enough to hold two coins at a time. If we put two pennies into the purse, the purse is totally filled with just two cents. But if we instead place two dimes in the purse, that same purse can hold 20 cents. When information is grouped into meaningful clusters, the brain can hold more content than if each fact is considered in isolation. We can apply this principle to help students master basic facts. For example, "one" is the identity element in multiplication, because multiplying a number times one yields a product that is the same as the original factor, as illustrated by the fact $7 \times 1 = 7$. Students who understand this concept can practice all the $\times 1$ facts simultaneously without overloading working memory, because although the student is practicing nine different facts, they are all examples of just one strategy.

Once an individualized set of facts has been identified, students need many opportunities to practice before mastery is achieved. Researchers have identified several characteristics common to effective practice activities. They include modeling so that the student can see both the fact and solution during practice (Codding et al., 2011; Riccomini, Stocker, & Morano, 2017), multiple opportunities to respond (Kubina & Yurich, 2012; McLeskey et al., 2017), immediate feedback (Fuchs et al, 2008; McLeskey et al., 2017), and an appropriate ratio of known to unknown facts. A ratio of nine known facts to one unknown fact has been found effective for mastering basic facts (Burns, 2005; Burns et al., 2019; Codding, Archer, & Connell, 2010; Riccomini et al., 2017). The IES Practice Guide recommends that students practice fact fluency for ten minutes at least three to four times per week (Gersten et al., 2009). Several techniques have been identified that incorporate these recommended characteristics and will enable teachers to provide effective, individualized fact practice. We will describe several of them here. Mastery practice activities should be used to develop fluency only after students have been exposed to the types of strategies introduced in the previous section. For example, at the beginning of the week the teacher might focus on a specific fact strategy. Once students understand the strategy, then math lessons later in the week could focus on mastering facts that fit that strategy, using one or more of the practice techniques described below.

Cover-Copy-Compare (CCC)

CCC is a practice activity that students complete independently. The technique has been used to practice math facts, vocabulary words, and other factual information for decades, and several versions of the technique have been developed (Becker, McLaughlin, Weber, & Gower, 2009; Grafman & Cates, 2010, Joseph et al., 2012; Poncy, Skinner, & Jaspers, 2006; Riccomini et al., 2017, Skinner, Turco, Beatty, & Rasavage, 1989). We describe the version proposed by Riccomini et al. (2017). The process consists of four steps. First, select facts for practice. The authors recommend using flashcards to assess the student's knowledge of the required math facts. Sort the facts into three piles: a FLUENT pile of facts that the student can answer in two seconds; a KNOWN pile that the student can answer accurately in three to five seconds, and an UNKNOWN pile of facts that students either answered incorrectly, needed to use a strategy to answer, or needed more than five seconds to answer. Next, create CCC practice sheets that contain nine known facts, one unknown fact and two to five facts from the fluent pile. The sheets should have a 9:1 ratio of known to unknown facts, with the unknown fact repeated throughout the sheet. The practice sheets should resemble the sheets in Figure 10.19, with the facts listed with answers provided, followed by two spaces where the student can write the fact.

Figure 10.19 CCC Practice Sheet

Cover-Copy-Compare Practice Sheet			
Name:			Date:
Fact		Write	Repeat
1. *(unknown)*	3 x 7 = 21		
2. *(known 1)*	3 x 4 = 12		
3. *(unknown)*	3 x 7 = 21		
4. *(known 2)*	3 x 3 = 9		
5. *(fluent 1)*	3 x 5 = 15		
6. *(unknown)*	3 x 7 = 21		
7. *(known 3)*	4 x 3 = 12		
8. *(known 4)*	2 x 8 = 16		
9. *(fluent 2)*	3 x 2 = 6		
10. *(unknown)*	3 x 7 = 21		
11. *(known 5)*	2 x 7 = 14		
12. *(known 6)*	3 x 6 = 24		
13. *(known 7)*	8 x 2 = 16		
14. *(fluent 3)*	3 x 1 = 3		
15. *(unknown)*	3 x 7 = 21		
16. *(known 8)*	6 x 3 = 34		
17. *(known 9)*	7 x 2 = 14		
18. *(fluent 4)*	3 x 0 = 0		
19. *(unknown)*	3 x 7 = 21		
20. *(fluent 5)*	2 x 3 = 6		

Once the practice sheets have been created, students must be taught how to use them. The procedure is simple. The student looks at the first fact and solution on the practice sheet, and then covers the fact and solution (this can be done by folding the paper over to hide the first column) and writes both the fact and the solution from memory in the first space provided on the answer sheet. Next, the student uncovers the fact and compares it to their response. If the fact is written correctly, the student moves on to the next fact on the sheet and repeats the process. If the fact is written incorrectly, the student crosses out the error, reviews the fact, covers it and tries again to write the fact and solution correctly. Continue to monitor the student's progress through weekly assessments and revise the lists of known, unknown, and fluent facts accordingly. Create new practice sheets that continue to use a ratio of nine known facts to one unknown fact, or experiment with different ratios of known to unknown facts if an individual student is ready for increased challenge. The CCC process is clearly modeled in a YouTube video (available at https://www.youtube.com/watch?v=QYU-70ajZhm). Intervention Central provides a template for creating CCC worksheets (available at https://www.interventioncentral.org/sites/default/files/pdfs/pdfs_interventions/ccc_worksheet_spelling_ sight_words_math_ horizontal.pdf) as well as an overview of the process.

Classwide Peer Tutoring

Peer tutoring has had mixed results in research studies (see the Best Evidence Encyclopedia report at www.bestevidence.org). These differences may be due to variations in the way peer tutoring is used. One approach that has received consistent positive results with struggling learners is Classwide Peer Tutoring (Fulk & King, 2001; Maheedy et al, 2003). This is a highly structured form of peer tutoring, and it should be implemented with fidelity in order to obtain optimal results. The teacher divides the students in the class into two teams. Within each team, pairs of students quiz each other on a pre-determined list of facts. To individualize practice, each student can have a personal set of flash cards containing the facts she needs to master, as well as a pile of mastered facts to review periodically. From these cards, the student should select one or two unfamiliar facts or fact clusters, plus additional review facts to make a total of five to ten cards. Next, students pair up and exchange cards. One student is the tutor and quizzes his partner on the partner's set of cards. Students are given scripted guidance on what to say when their partner responds correctly, and how to respond when their partner makes a mistake. If the answer is correct, the tutor awards two points. If the answer is incorrect, the tutor helps the tutee identify the fact. The tutee practices the fact three times, and earns one point for successful practice. Then the card is placed a few cards back in the pile so it can be reviewed again. After five minutes, the teacher has the pairs reverse roles. At the end of the ten-minute practice period, pairs total their points, and those points are added to their team's score for the day. The game allows everyone in the group to be involved in the same activity, but practice is differentiated so that each person can practice an individualized set of facts. For a brief online demonstration of Classwide Peer Tutoring, see https://www.youtube.com/watch?v=V9i5yWzz79s.

Incremental Rehearsal

Incremental rehearsal is another flashcard technique that follows a process similar to Classwide Peer Tutoring, except that it can be implemented with either a teacher, peer tutor, parent, or other volunteer, and is typically completed orally (Burns et al., 2019; Kupzyk et al., 2011). The student is pre-tested in a manner similar to what we described for CCC, and a set of facts is created that contains a ratio of one new or unknown fact to seven, eight or nine known facts. During the practice session, the teacher or peer tutor shows the first fact, says it aloud and reads the answer. The student then repeats both the fact and the answer two times. After that, they work through the deck of flashcards. If the student answers incorrectly or takes longer than three seconds to respond, the teacher or tutor reads the fact and answer, and then the student repeats both the fact and the answer. The student continues to respond to each of the flashcards in the deck until he can answer all of them in two seconds or less without error. Once the student has mastered all the cards in the deck, a new flashcard is drawn from the pile of unknown facts and introduced as described above. One of the known facts is removed from the deck, and the process is repeated. Information on implementing incremental rehearsal is available from Intervention Central (https://www.interventioncentral.org/academic-interventions/math-facts/math-computation-promote-mastery-math-facts-through-incremental-re), and a video example is available on YouTube (https://www.youtube.com/watch?v=ke-4HETehE6Q).

Taped Problems

Taped problems provide another opportunity for students to practice facts independently. Once again students are pretested, and the teacher creates a worksheet containing known and unknown facts, and also creates an audio recording of the same facts and answers, with a constant time delay between each fact (McCallum et al., 2004). To practice, the student listens to the recording and tries to "beat the tape" by writing the answer to each fact before the teacher says it on the recording. Taped problems can be adapted by creating initial tapes with longer delays between problems, perhaps 4-5 seconds, and gradually decreasing the time so that students have to respond more quickly on subsequent trials (Cooper et al., 2007). Additional information about taped problems is available from the Evidence Base Intervention Network (http://www.interventionexpress.com/uploads/1/6/8/5/16851140/taped-problems.pdf), and a video illustrating the process can be found at https://www.youtube.com/watch?v=f7xum6xZOtE.

Computer-Based Instruction (CBI)

A multitude of computer programs and apps advertise that they will help students master basic facts. Efficient drill activities provide targeted practice in just a limited number of unfamiliar facts or a particular strategy, coupled with periodic review of any previously mastered facts. Programs that provide customized drill matched to individual student needs can facilitate fluent fact retrieval. Some programs allow teachers to individualize the activity to include only those facts a particular student needs to practice, and other programs assess student mastery and generate an individualized list of facts to practice. Such programs can be very effective. Unfortunately, many computerized programs offer random practice of too many facts, such as those that practice all the 100 addition facts or multiplication facts. While students may enjoy spending time on the computer, providing such brief exposure to a large number of facts is unlikely to build automaticity. Hawkins et al. (2017) identified characteristics found in CBI programs that effectively develop automaticity with basic facts. They include: (1) customization features that allow the instructor to individualize practice, (2) ample opportunities to respond, (3) immediate feedback and error correction, and (4) progressive monitoring features. Intervention time is valuable, and instructors must choose wisely to provide the type of focused practice that will help students develop computational fluency.

Additional Ideas for Providing Mastery Practice

Caution is needed when selecting practice activities. Many of the activities that teachers have traditionally used to develop automaticity can be counterproductive. For example, in the popular game, "Around the World" or "Traveler," the whole class watches while two students compete to see who can be the first to call out the answer to a math fact. Students who have already mastered most of the facts may find this game enjoyable, but those who most need the practice may find it humiliating. Public embarrassment promotes math anxiety, and is not a productive way to increase mathematical competence.

Fact practice should be individualized to allow each student to focus on the specific facts he or she needs to learn, and limit the number of unfamiliar facts introduced in each session. The IES practice guide suggests that students should focus on just two unfamiliar facts at a time, coupled with ongoing review of previously mastered facts (Gersten et al., 2009). Interventions are typically provided in small-group settings, and it is rare that all students in the group should be focusing on the exact same set of facts or fact clusters. To achieve optimal learning outcomes, interventionists need to differentiate practice activities. In addition to the techniques described above, a variety other formats can be used to provide individualized practice.

1. **Worksheets:** Students generally view completing worksheets as a rather dull activity, so worksheets should be used sparingly. However, occasional worksheet practice can be beneficial, and it is easy to provide individualized worksheets that focus on a particular fact or fact cluster. The worksheets illustrated in Figures 10.12 and 10.18 are a good example. Everyone in the group could be completing a fact worksheet, but the sheets can be individualized to allow each student to practice just the one or two unfamiliar facts or fact clusters he needs to master.
2. **Board Games:** Almost any board game can be used to practice math facts by simply providing a stack of fact flash cards and requiring students to correctly solve a fact before advancing their game piece. Commercial board games designed for practicing math facts generally provide a stack of flash cards. The cards are placed in the middle of the board, and every student draws from that same pile of cards. However, if the facts do not match each student's individual needs, then little learning may occur. Instead, allow each student to use her own set of flash cards, selected as described previously. The individualized flash cards allow each student to experience differentiated practice on the specific facts she needs to master.
3. **Card Games:** Students enjoy playing games, and their increased engagement and motivation can facilitate learning. Card games are easy to differentiate. All the students in the class can learn to play a basic card game like the Make-a-Pair or Concentration games described in the online materials. When it is time to practice math facts, students working on the same sets of facts can be grouped together and given a deck of cards that contains only the facts those students need to practice. All students in the class can be playing the same card game, but if their cards are differentiated, then they are receiving the type of targeted practice that has been shown to maximize learning outcomes.
4. **Stations:** Learning stations can be adapted to provide differentiated practice in basic math facts (Forbringer & Fahsl, 2007, 2009). They are especially effective in the regular classroom where teachers need to differentiate practice for large groups of students. When using differentiated stations, students are grouped homogeneously. All the students in a single group need to work on a similar set of facts. Before a group of students enters the station, the activity is adjusted to focus on just the facts the students in that group need to learn or review. For example, a class might contain one group of students

who are working on adding doubles, and another group who are multiplying by fives. One of the stations could contain the egg carton game described in the online materials. In this game, a numeral from 0 to 9 is written inside each cup of an empty egg carton. Students take turns rolling or dropping a small ball into the egg carton, and the number written in the cup where their ball lands is used in a fact calculation. Students compete to see who will be the first to accumulate a pre-determined number of points. In the group practicing doubles, students would double the number where their ball lands and use that as their score for the round. Those working on the five tables could multiply the number by five, while a third group of students who are working on counting-on could add one to the number in the cup. Each group would play the same egg carton game, but the activity would be differentiated to provide targeted practice in the facts those students need to master. A second station in the same classroom might contain a board game. Before each group enters the station, the teacher would switch the deck of cards to provide just the facts that group of students needs to practice. Each group would play the same board game, but the teacher would differentiate the activity by providing a deck of cards that matches the needs of the students in the group. When using differentiated stations, it is advisable to schedule station time so the teacher has time to adjust the materials before each group's arrival at a station. Instead of having students rotate through all the stations in a single period, they can rotate over the course of several days. For example, if there are five stations, then students could complete a different station each day, and in a week everyone will go through every station. This allows the teacher time after school to switch materials at each station, so when the students arrive the station is prepared with materials appropriate for the students scheduled to use the station that day. The activity itself will look the same, so students may not even realize that the facts they practice are differentiated.

5. **Self-Monitoring:** Self-monitoring is a strategy that involves students monitoring their own behavior and recording the results. As we discussed in Chapter 4, studies have shown that students who use self-monitoring are more engaged and more productive, have greater accuracy, and show increased awareness of their own behavior (Carr, 2014; Falkenberg & Barbetta, 2013; Schulze, 2016, McLeskey et al., 2017). Math facts are an ideal place to use self-monitoring, because students can easily chart their progress mastering the basic facts. The IES practice guide suggests, "Allow students to chart their progress and to set goals for improvement" (Gersten et al., 2009, p. 46).
6. **Rewards:** In Chapter 4, we also discussed the value of providing rewards. Rewards have proven so valuable in increasing achievement that the IES Practice Guide recommends including them in all interventions: "Tier 2 and Tier 3 interventions should include components that promote student effort *(engagement-contingent rewards)*, persistence *(completion-contingent rewards)*, and achievement *(performance-contingent rewards)*" (Gersten et al., 2009, p. 44). Fact mastery provides an excellent opportunity for offering rewards. Follow the guidelines we described in Chapter 4 for designing incentive systems. For example, instead of offering a big reward when the student masters all the addition facts, design the system to reward improvement. Every new fact mastered is cause for celebration.

Summary

Computational fluency is an essential component of mathematical proficiency. Students are expected to have mastered addition and subtraction facts by the end of second grade, and multiplication and division facts by the end of fourth grade. Individuals who meet these benchmarks can focus their attention on problem solving and higher-level computational procedures, while those who must still focus on the process of computing basic facts will have less cognitive capacity available for complex mathematical procedures. Because fact fluency is such an important skill, the IES practice guide recommends that ten minutes of each intervention session focus on developing fluency with basic facts (Gersten et al., 2009).

Fluency develops in phases. In Phase One, students develop an understanding of basic operations. Students in this phase typically model problems and count to find the answer. We described methods for developing this basic conceptual understanding in the previous two chapters. In Phase Two, students learn strategies for solving basic fact problems. In this chapter, we described a variety of strategies to help students solve addition, subtraction, multiplication, and division fact problems. Once a student understands and can apply a particular strategy, then he is ready to focus on automaticity, which is Phase Three. The goal of phase three is that students reach mastery, which means they can compute basic facts in less than three seconds. Researchers have identified several characteristics of effective mastery practice activities. They include modeling the fact, providing multiple opportunities to respond followed by immediate feedback, and providing an appropriate ratio of known to unknown facts. In this chapter, we described several techniques interventionists can easily implement that incorporate these evidence-based characteristics.

Experts recommend that instruction during tiered interventions should focus on foundational concepts, including whole numbers, rational numbers, and problem-solving. In the past several chapters, we have discussed several ways to help students develop number sense and master operations with whole numbers. In the next chapter, we turn our attention to rational numbers.

11

Representing Rational Numbers

In the early grades, interventions for students receiving Tier 2 and Tier 3 support should emphasize counting, number value, place value, and operations with whole numbers, as discussed in Chapters 7-10. Once students have mastered this content, the focus shifts to rational numbers, including understanding the meaning of fractions, decimals, ratios and percent, and operations using rational numbers (Gersten et al., 2009). These topics represent foundational proficiencies and are pre-requisites for further mathematical progress.

While many of the methods for introducing rational numbers during interventions are similar to those used in the core curriculum, interventions differ in two important ways. First, instruction during interventions should be explicit and systematic (Gersten et al., 2009; McLeskey et al., 2017). This includes making sure pre-requisite knowledge and skills have been mastered and are reviewed before new content is introduced, modeling skills and strategies before asking students to perform them, and providing carefully sequenced guided and independent practice, as described in Chapter 5. Second, interventionists need to introduce new content using concrete and pictorial representation, explicitly link the representation systems, and make sure students fully understand the concepts using visual representations before relying solely on abstract symbols and numbers (Bouck, Park & Nickell, 2017; Gersten et al., 2009; Gibbs, Hinton, & Flores, 2017). While concrete and pictorial representation is used in Tier 1, core materials often move too quickly to abstract representation before learners who struggle with mathematics are able to fully grasp the concepts (Gersten et al., 2009; van Garderen, Scheuermann, Poch, & Murray, 2018). In this chapter, we will discuss ways to incorporate these intensive intervention strategies when introducing rational numbers.

Fractions

Developing Fraction Concepts

Fractions present one of the greatest challenges students encounter. National and international test results reveal that American students have consistently struggled with basic fraction concepts (NMAP, 2008, 2019; Siegler, 2017). Understanding fraction concepts is necessary to perform meaningful computations with fractions, and fractions are a pre-requisite for decimals, percent, ratio and proportion, and algebra. Knowledge of fractions in fifth grade predicts student's math achievement in high school, even after controlling for the student's IQ, knowledge of whole numbers, and family education level or income (Siegler, 2017). Even students who have not experienced previous mathematical difficulty can be challenged by fractions. For students with a history of mathematical difficulty, the problem is magnified.

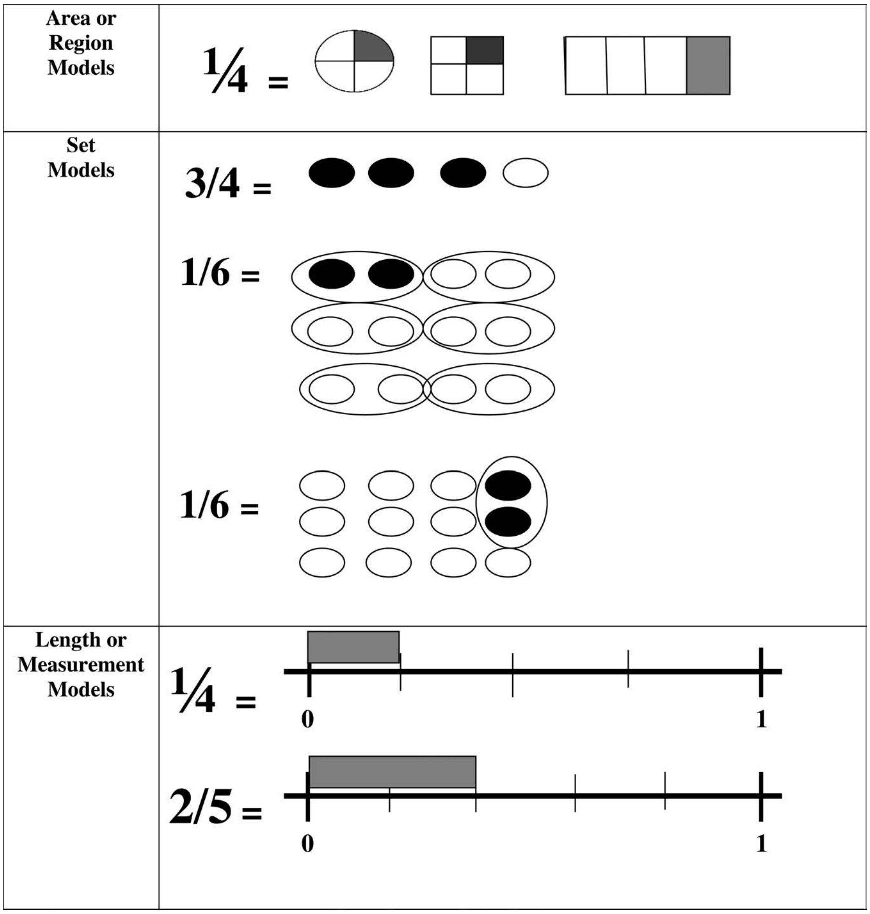
To understand fractions, students must master a few big ideas. First, fractional parts are formed when a whole or unit is divided into equal parts. In other words, to understand a fraction, students first need to identify the unit and then make sure it is divided into equal parts. Students who struggle with fractions sometimes miss the importance of having equal parts. The concept of *unit* can also confuse students, because the word has several different mathematical applications. The smallest piece in base-ten blocks is sometimes called a *unit block*. For fractions, the *unit* is the whole object, set, or length that is divided into equal parts. For example, if one pizza is cut into six pieces, the whole pizza is the unit. If a dozen cookies are shared equally among three friends, then the *unit* is the original set of 12 cookies. Another term, *unit fraction,* is used to describe one piece of the unit, i.e. any fraction with a numerator of one. In the pizza example, the whole pizza was the unit, but the unit fraction is 1/6 of the pizza, because the pizza was cut into six pieces. Later, students will see another application of the word *unit* when they begin to work with multiplicative comparison problems in fourth grade. Because the word *unit* is used in a variety of ways in mathematics, interventionists should explicitly teach this mathematical term as thoroughly as they would teach any new vocabulary.

The third big idea that students need to understand is how fractions are labeled. They need to know that the denominator tells how many equal parts are in the unit and the numerator tells how many of those parts we have. Models play an important role in helping students understand these big ideas. Figure 11.1 provides examples of three different types of models used to illustrate basic fraction concepts: (1) area models, (2) set models, and (3) measurement models.

Area models involve dividing one whole object into equal parts. Area models are generally the easiest fraction models for students to understand. A variety of materials are available to help students learn to create area models of fractions, but the most commonly used manipulative is the fraction circle. When cut like a pizza pie, fraction circles provide an excellent, concrete way to help students understand the relative value of fractional parts of wholes. Unfortunately, too often fraction circles are the only type of fractional representation that students encounter. NCTM (2000) recommends that all students experience multiple representations of mathematical concepts and have the opportunity to translate among representations, because connecting one form of representation to another enhances understanding. Modeling fractions with other shapes, such as squares, rectangles, and triangles, can help students develop a more solid understanding (see Figure 11.1). Manipulatives like pattern blocks, Cuisenaire rods and geoboards can also be used to model fractions. Fraction towers are especially useful, because the pieces snap together like the pop cubes used for counting whole numbers and so are less likely to be jostled out of place than some other manipulatives. See Figure 11.2 for an illustration of these manipulatives.

Once students are comfortable using area models to represent fractional parts of whole numbers, they need experiences with other types of models. Set models present a greater challenge, because when students see a set of objects, they tend to find it harder to identify what constitutes the unit. Sets vary in size; they may contain two items or 2000. For example, a set might be a dozen eggs, all the students in the classroom, or a box of crayons. Whatever its size, the entire collection of items in the set forms the unit, which is counted

Figure 11.1 Modeling Fractions



as one whole. When we divide a set into parts, each of the fractional parts is a subset of the unit. This is illustrated in the first example of set models shown in Figure 11.1. The example depicts a set of four counters, three of which are shaded. In this example, the four counters form the unit. Each individual counter represents one subset or fractional part, so three out of four counters, or 3/4 of the counters, are shaded. In the second and third examples of set models, the unit consists of 12 counters. The unit is divided into six groups or subsets, each of which contains two counters. One subset contains shaded counters, so it represents 1/6 of the unit. Because the number of objects forming the unit and its subsets varies from one set model to another, students sometimes find set models confusing. For this reason, the Common Core State Standards for kindergarten are limited to dividing a whole object (circle or square) into simple unit fractions (halves and fourths). Fractions with numerators other than one, and fractions formed by dividing other objects or sets, do not appear in the standards until third grade.

Fractions can also show measurement, as seen in rulers and number lines. In a measurement model, the unit is the distance from 0 to 1, and the space between subdivisions represents the fractional parts. When we introduce rational numbers using fraction circles and bars, students count the number of pieces. On a number line, they count spaces. For this reason, locating fractions on a number line challenges many learners, and individuals who struggle with mathematics will need explicit instruction if they are to succeed with this method of representation. The effort is worth it, however, because a number line can

Figure 11.2 Concrete Representations of Fractions

Modeling 1/2 with a Geoboard

Modeling 1 whole and ½ with Pattern Bocks

Modeling 1 whole, ½, 2/4, 4/8
with Cuisenaire Rods

Modeling 1 whole, 1/2, 2/4
with Fraction Towers

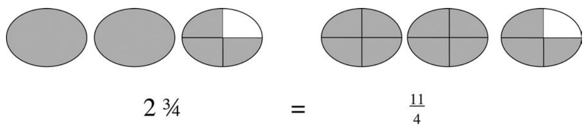
be used to represent any fraction, decimal, or percent. When students use fraction circles or fraction bars to represent fractional parts of a whole, they draw a different model for each different denominator. A number line provides a consistent model that students can use to represent any quantity, so once students develop a mental number line they can easily compare the values of any rational numbers.

Because students generally find it more difficult to master set and measurement models, it is helpful to wait to introduce these concepts until after the student has demonstrated mastery of area models. Transitioning too quickly from area models to set and measurement models can lead to confusion. For a thorough discussion of the use of models to develop fraction concepts, see *Elementary and Middle School Mathematics: Teaching Developmentally* (Van de Walle et al., 2019).

Using the CPA Sequence with More Advanced Fraction Skills

While most programs use some form of concrete or visual representation to introduce the concept of fractions, few programs follow the CPA continuum when introducing more complex skills such as converting mixed numbers to improper fractions, or adding, subtracting,

Figure 11.3 Mixed Numbers and Equivalent Fractions

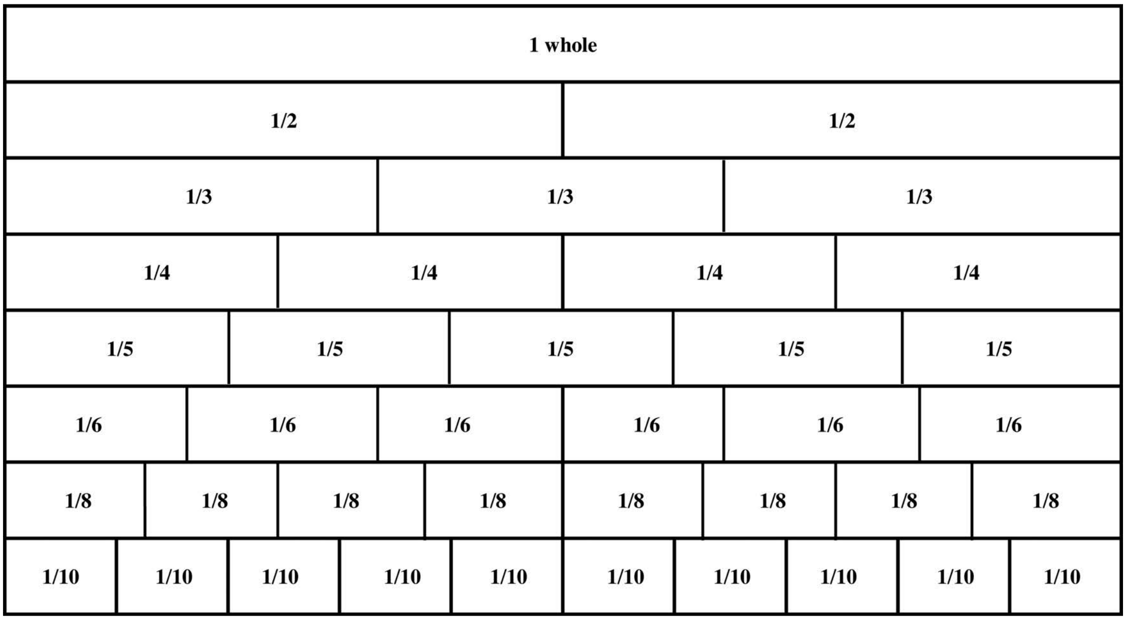


multiplying, and dividing fractions. In one study, researchers examined three middle-school textbook series to determine how well they incorporated representations into their lessons on fractions (Hodges, Cady, and Collins, 2008). In the texts they examined, the use of concrete representation ranged from a mere 0.25 percent up to a high of only 5.12 percent. Visual representations of fractions appeared between 7.28 percent and 27.31 percent of the time, while the vast majority of lessons relied only on abstract words and symbols. If these findings are typical, then it is not surprising that American students are struggling with fractions. The lack of concrete and pictorial representation in middle school textbooks suggests that many students will have only abstract experiences with fractions. When skills are introduced using only abstract symbols or words, students often memorize rote procedures without fully understanding what they are doing. For example, core materials sometimes use a totally abstract process when teaching students to convert mixed numbers to their fractional equivalents. To express 2¾ as a fraction, the teacher might say, "First, multiply the whole number times the denominator, then add the numerator. In this problem, 2 x 4 equals 8, plus 3 makes 11. That's your numerator. Keep the denominator the same. So, 2¾ = 11/4." This rote procedure is not meaningful. Meaningful information is more easily remembered (Wolfe, 2010), so students who use concrete and visual representations to develop an understanding of the underlying concepts are also more likely to remember and be able to apply their knowledge in the future. Figure 11.3 shows how a model can help students understand the relationship between fractions and mixed numbers. In this example, the mixed number 2¾ is shown using two whole circles, plus three of another circle. Building on students' previous understanding of equal shares, we can partition the two whole circles into fourths, so that all the circles are divided into equal-size pieces. Students can see that when they cut each of the two whole circles into four equal parts, they end up with eight parts. This is the $2 \times 4 = 8$ mentioned in the abstract explanation, now given meaning through the visual representation. The model also contains three additional parts. When we count all the shaded parts, we have a total of 11 parts ($8 + 3 = 11$). We did not change the size of the individual parts. They are still fourths, so the denominator stays the same. Students can see that 2¾ is the same as 11/4. When the abstract explanations are meaningfully associated with a concrete or pictorial representation, the abstract procedure is more easily understood and retained. Even if they forget the abstract steps, students who understand the underlying concepts can figure out the solution by creating a quick sketch. Fractions less than one used to be called "proper fractions," and fractions representing quantities greater than one were called "improper fractions." Note that in the previous example, we did not use the term "improper fraction" when we mentioned 11/4. The term "improper" implies that something is wrong with these fractions. There is nothing mathematically incorrect about a fraction equal to or greater than one, and so the term "improper fractions" does not appear in the CCSSM.

Equivalent Fractions

Fraction equivalence is another big idea in understanding fractions. Students need a variety of concrete experiences in order to understand that the same fractional portion of a whole or set can be expressed using different symbolic representations. For example, the same

Figure 11.4 Equivalent Fraction Strips



quantity can be labeled as 1/2, 2/4, 3/6, and so on. To help students recognize that the same fractional portion can be created using different fraction pieces, we can ask them to begin with a fraction piece that represents a familiar fraction such as 1/2 or 1/3, and use fraction manipulatives to find as many single-fraction names for the area as possible. For example, they could use fraction circles to illustrate 1/2 and then try laying other fractions on top of the 1/2 model to determine which can be used to cover 1/2 exactly. This provides a concrete model that helps students understand that 1/2 is the same amount as 2/4 or 4/8. Equivalent fractions can also be modeled using strips of paper that have been folded to represent different fractions. An unfolded strip would represent one whole, and other strips of the same length could be folded in half, thirds, fourths, and so on. When the strips are laid out below each other, fractions that are equivalent are readily apparent. See Figure 11.4.

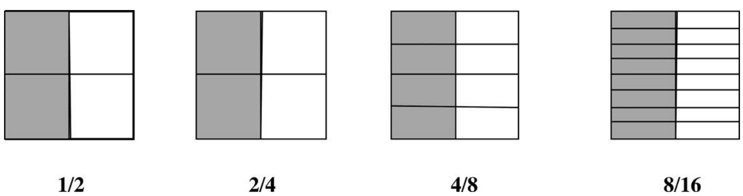
Length models can also be used to develop the concept of equivalence. Fraction bars, towers, or pop cubes are all examples of length models. The cubes are especially well suited to modeling equivalent fractions because the plastic cubes snap together to form sturdy towers. Once they are snapped together, the pieces of adjacent towers cannot slide around as fraction circles or strips of paper sometimes do, and the stability makes it easier for students to identify fractions that are truly equivalent. These same factors make this manipulative effective when students are comparing fractions. See Figure 11.5. Fraction towers can be laid side by side, so it is possible to compare several sizes at once to see which combinations are equivalent. In contrast, fraction circles must be laid on top of each other, which obscures the bottom pieces and so makes it more difficult to compare multiple sizes simultaneously.

Equivalent fractions can also be represented using an area model of a simple square, with vertical and horizontal lines drawn to create fractional pieces. Figure 11.6 shows a square shaded to represent ½. Horizontal lines on the square can separate the region into equal parts, which provides a quick way to model equivalent fractions. The second, third, and fourth squares in the figure show what happens when a region is cut into four, eight, or sixteen equal parts. Students can easily sketch squares on paper, so they can use this representation independently, even when they do not have access to manipulatives.

Figure 11.5 Using Fraction Towers to Compare Fractions

Comparing 1 whole, 1/2 and 2/4

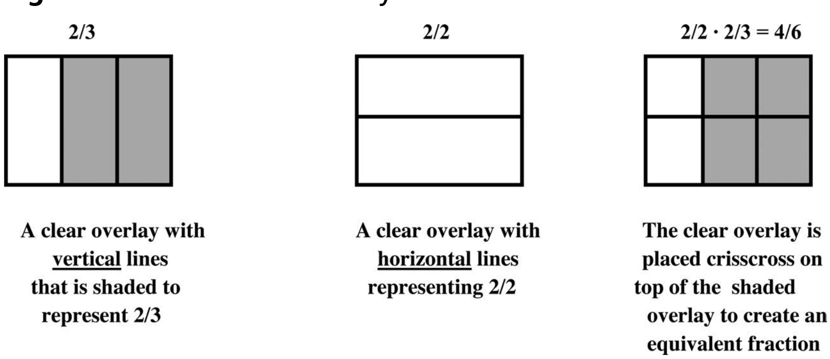
Figure 11.6 Fraction Squares



Fraction overlays are a three-dimensional version of the squares described above. The overlays are cut from overhead transparency sheets or other clear plastic sheets, with lines drawn on each square in permanent marker to create halves, thirds, and so on. By shading parts of the square with a dry-erase marker, students can represent specific fractions. Just as we modeled with the squares in Figure 11.6, students can turn an unshaded overlay on its side so that the lines run horizontally, and then place it on top of the shaded overlay to create an equivalent fraction. See Figure 11.7.

After students have had multiple experiences using concrete and pictorial representation to model equivalent fractions, they will eventually need to learn the equivalence algorithm so that they will be able to perform operations with fractions without needing to rely on visual models. The algorithm for finding equivalent fractions is this: multiply or divide both

Figure 11.7 Fraction Overlays



the numerator and denominator by the same non-zero number. This symbolic procedure for finding equivalent fractions is based on the identity property of multiplication, which says that multiplying a number by one does not change the number. Therefore, we can multiply any number by a fraction equivalent to 1 (e.g. 2/2, 3/3, or 4/4) without changing its value. Consider these examples:

$$\left(\frac{2}{2}\right)\frac{2}{3}=\frac{4}{6} \quad \left(\frac{3}{3}\right)\frac{3}{4}=\frac{9}{12}$$

Instead of emphasizing the meaning of equivalent fractions, teachers sometimes introduce equivalent fractions using a totally abstract process. For example, to find how many sixths are equivalent to 2/3, they might say, "What do you multiply times 3 to get 6? Two, right. So let's multiply both the numerator and the denominator by two: $2 \times 2 = 4$, and $2 \times 3 = 6$. So our answer is 4/6. 2/3 = 4/6. To find equivalent fractions, just multiply the numerator and the denominator by the same number." This abstract procedure may seem easy, but it is not meaningful. Sketched squares or fraction overlays are an excellent way to give meaning to the algorithm, because when we turn a clear overlay horizontally and lay it on top of a shaded overlay, we are actually multiplying the fraction by an equivalent of 1. Figure 11.7 shows that when a clear overlay representing 2/2 is placed over the square shaded to represent 2/3, it creates 4/6. In other words, $(2/2)2/3 = 4/6$. Note that we use fraction squares, rather than fraction circles, to illustrate equivalence. The procedure does not work with circles, because the resulting pieces will not be of equal sizes. Early experiences with equivalence should focus on concrete and pictorial models. The abstract algorithm involves multiplying fractions, so it should not be taught until students begin multiplying fractions.

Students need many experiences manipulating concrete and pictorial models of equivalent fractions before teachers introduce the abstract algorithm. Explicitly linking the algorithm to students' experiences with concrete and pictorial representations will help them understand the underlying meaning, and make it more likely that they will remember the algorithm and apply it successfully in the future. Many textbooks, especially at the middle-school level, rely on a strictly symbolic approach to teach equivalent fractions. Students are taught the algorithm without fully understanding its significance, and as a result may forget it, confuse it, or execute the procedure by rote without being able to apply it meaningfully in problem-solving situations. If the available program does not emphasize concrete and pictorial models, then interventionists can add this component to their lessons to intensify instruction.

Adding and Subtracting Fractions

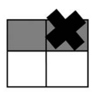
Concrete and pictorial representations are equally useful when introducing addition and subtraction of fractions. When students first learn these operations, the most frequent error they make is to add or subtract the denominator along with the numerator. Given the fractions 1/4 + 2/4, they may mistakenly calculate the sum as 3/8. This error reveals a lack of conceptual understanding. If students first use manipulatives to solve this problem, as shown in Figure 11.8, the error is far less likely because they can see that the size of the pieces has not changed, so therefore the denominators should not change either.

Once students are comfortable adding and subtracting fractions with like denominators, they can tackle addition and subtraction of mixed numbers with like denominators. Adding and subtracting mixed numbers is not conceptually difficult, but it involves multiple steps. Although there are several possible methods for adding and subtracting mixed numbers, perhaps the simplest is to first convert all the mixed numbers to equivalent fractions and

Figure 11.8 Adding and Subtracting Fractions

Adding and Subtracting Fractions

1/4 + 2/4 = 3/4



2/4 - 1/4 =1/4

then add or subtract. Once the calculation is complete, the resulting fraction can be converted back to a mixed number. Students who have deficits in executive functioning are likely to stumble because of the number of steps required. To support students in this process, interventionists can intensify instruction by providing a list of skill steps and encouraging students to check off each step as they complete it. See Figures 11.9 and 11.10 for examples.

Figure 11.9 Adding Mixed Numbers with Like Denominators

Add Mixed Numbers with Like Denominators

1. Change the mixed numbers to equivalent fractions.
2. Add.
3. If the result is a fraction greater than 1, convert it to a mixed number.
4. Record

Example

2 2/3		7/3		
+ 1 1/3	⇨	+ 5/3		
		12/3	⇨	4

Representation

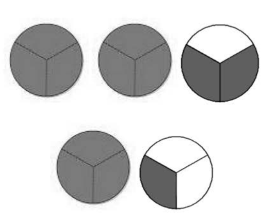


Figure 11.10 Subtracting Mixed Numbers with Like Denominators

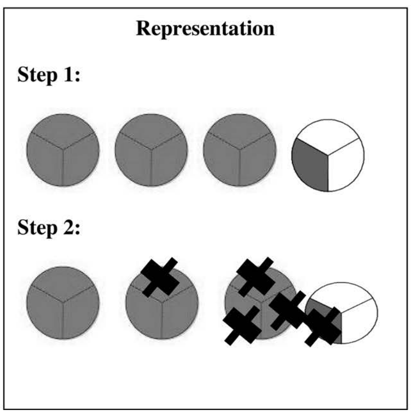
Subtracting Mixed Numbers with Like Denominators

1. Change the mixed numbers to equivalent fractions.
2. Subtract.
3. If the result is a fraction greater than 1, convert it to a mixed number.
4. Record

Example

3 1/3 ⇨ **10/3**
- 1 2/3 ⇨ **- 5/3**

5/3 ⇨ **1 2/3**



Once students can find equivalent fractions and add and subtract mixed numbers with like denominators, they are ready to tackle addition and subtraction of mixed numbers with unlike denominators. This procedure also involves multiple steps, so again interventionists can support students by providing a list of steps and encouraging students to monitor their progress. Figure 11.11 shows a list of suggested skill steps for adding and subtracting fractions and mixed numbers with unlike denominators.

Step one of this procedure is to rewrite the problem using a common denominator. The simplest, most straightforward way to find a common denominator is to multiply the denominators provided; the resulting product represents a common denominator. For example, to find a common denominator for 2/3 + 1/4, we multiply 3 × 4 to obtain the common denominator of 12. This process is identical to the process for crisscrossing fraction overlays to find equivalent fractions that we described previously. If we first model one fraction with vertical lines defining the pieces, and then place an overlay on top that uses horizontal lines to model the denominator of the other fraction, we have effectively created the common denominator. In other words, if we crisscross a clear overlay cut into fourths on top of a square that illustrates 2/3, we have shown that 2/3 = 8/12. If we repeat the process by crisscrossing a clear overlay cut into thirds on top of a square that illustrates 1/4, we have shown that 1/4 = 3/12, and have successfully rewritten the problem using common denominators.

Figure 11.11 Adding and Subtracting with Unlike Denominators

Skill Steps:
Adding & Subtracting Fractions

1. Rewrite with common denominators.
2. Add or subtract.
3. Rewrite the answer in lowest terms, if needed.

Skill Steps:
Adding & Subtracting Mixed Numbers

1. Rewrite with common denominators.
2. Change mixed numbers to equivalent fractions.
3. Add or subtract.
4. Rewrite the answer in lowest terms, if needed.

Example #1: Adding Fractions

2/3 + 1/2	⇨	4/6 +3/6	⇨	4/6 +3/6 7/6	⇨	4/6 +3/6 1 1/6

Example #2: Subtracting Fractions

4/5 - 1/3	⇨	12/15 - 5/15	⇨	12/15 - 5/15 7/15

Example #3: Adding Mixed Numbers

1 2/3 + 2 1/2	⇨	1 4/6 + 2 3/6	⇨	10/6 + 15/6	⇨	10/6 + 15/6 25/6	⇨	4 1/6

Example #4: Subtracting Mixed Numbers

4 1/3 - 1 1/2	⇨	4 2/6 - 1 3/6	⇨	26/6 - 15/6	⇨	26/6 - 9/6 17/6	⇨	2 5/6

For some students, it may be best to continue to use the common denominator formed by multiplying denominators and not bother to introduce the idea of lowest common denominator. Math teachers often spend a great deal of time teaching students to find lowest common denominator, but students who have not mastered basic facts struggle with this process. Interventionists may consider omitting it, because while learning to find a common denominator is necessary, using the lowest common denominator is not an essential skill. Multiplying denominators produces a correct answer, and instructional time may be more effectively spent on other topics. However, when the denominators being multiplied contain multi-digit numbers, the resulting common denominator can be very large and unwieldy. For this reason, the interventionist may choose to introduce least common denominators. Instead of teaching students to use factor trees, we have found a shortcut method that is effective. Steps for the shortcut method are listed in Figure 11.12. Using the shortcut method to find a common denominator for the fractions 4/6 and 5/12, we would first examine the two denominators. The numeral 12 represents a larger quantity than the numeral 6, so we will attempt to divide 12 by 6. (Note that we are focusing on the size of the numeral in the denominator, not the size of the fraction. Sixths are larger fraction pieces than twelfths, but we are looking at the numerals 6 and 12, and 12 represents a larger

Figure 11.12 The Shortcut Method for Finding Lowest Common Denominators (LCD)

The Shortcut Method for Finding Lowest Common Denominators (LCD)

1. Look at the denominators. Can the larger numeral be divided evenly by the smaller denominator?
2. If YES, then the larger numeral is the LCD.
3. If NO, double the larger numeral and try again.
4. Continue to triple, quadruple, etc. until you find the LCD.

quantity than 6.) Step 2 of the procedure asks whether the larger numeral can be divided evenly by the smaller denominator. In this case, the answer is yes: $12 \div 6 = 2$. Since 12 can be evenly divided by 6, we know that 12 is the lowest common denominator, and so we would express both fractions as twelfths. However, if the answer to Step 2 is no, then we would proceed to step 3. For example, to find the lowest common denominator for the fractions 1/6 and 1/9, we would identify 9 as the larger numeral. Since 9 is not evenly divided by 6, we would proceed to step 3 and try doubling 9. The resulting answer of 18 can be evenly divided by 6, so 18 is the lowest common denominator. The fractions 1/3 and 1/4 illustrate a problem that requires tripling to find the lowest common denominator. Four is not evenly divisible by 3, and if we double it, we find that 8 is not evenly divisible by 3 either. However, if we triple 4 to obtain 12, we have found a number that can be evenly divided by 3, and so we have identified the lowest common denominator. Students generally find the shortcut process fairly simple, and it is easily modeled using fraction overlays. If we first model the fraction whose denominator contains the larger numeral and then crisscross a clear overlay depicting 2/2 on top of the model, we have illustrated the doubling process. If we crisscross 3/3 on top of the fraction, we have modeled tripling the number. The shortcut method may enable students to quickly identify lowest common denominators when working with large numbers.

Adding and subtracting fractions challenges students because of the many steps involved. Students with executive processing problems or short-term memory deficits may become confused. As with all topics we have discussed, when providing interventions, it is important to first model the process with concrete and pictorial representation before introducing abstract procedures, and also to provide a clear list of steps for students to follow. This scaffolded support can be gradually faded as students gain confidence and proficiency.

Core materials often stress the importance of reducing fractions to lowest terms, but interventionists should use caution here, too. Although many teachers insist that fractions must always be written in lowest terms, the answer is mathematically correct whether or not it is reduced. Therefore, interventionists should consider whether a student should be required to state the answer in lowest terms. Reducing fractions adds a step to the computational process, and students who have limited working memory may be overwhelmed by this additional step. Follow the principles of systematic instruction and break instruction into smaller pieces to reduce the cognitive load. First introduce the process of adding and subtracting fractions without requiring students to reduce the answer to lowest terms. Once students are comfortable with this process, then introduce reducing. Introduce skills in small, carefully sequenced steps, and allow students sufficient time to master each step before moving forward.

Multiplying Fractions

Multiplying fractions builds on students' previous knowledge of multiplying whole numbers. According to the Common Core State Standards for Mathematics, in fourth grade students learn to multiply a fraction by a whole number and relate this process to multiplication of whole numbers. In other words, they understand that, if 3×4 indicates 3 sets of 4, then $3 \times 1/4$ indicates 3 sets of 1/4. Figure 11.13 shows the similarity between modeling multiplication of whole numbers and multiplication of fractions.

Multiplication is repeated addition, and the models in Figure 11.13 clearly reflect the connection between multiplication and addition. Some multiplication problems can be represented using the same fraction circles and fraction towers students used to model addition. See Figure 11.14.

Although fraction circles work well when one factor is a multiple of the other fraction, when the factors are not multiples of each other, fraction circles are not effective. For example, the fraction circles would not work well to illustrate a problem like $1/4 \times 2/3$. In this case, a pan of brownies provides a simple but meaningful concrete experience to model the process. Let's say I bake a pan of brownies to take to my students. To my dismay I discover

Figure 11.13 Modeling Multiplication

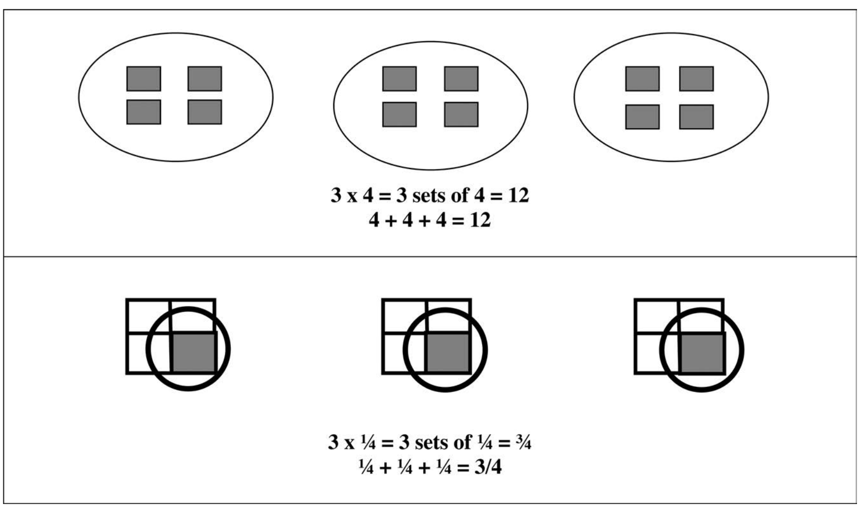
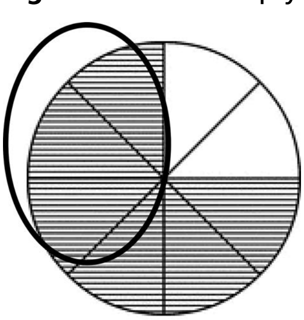


Figure 11.14 Multiplying by a Fraction



1/2 x 6/8 = 3/8
or
1/2 of 6/8 = 3/8

Figure 11.15 Modeling Multiplication

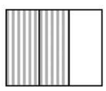
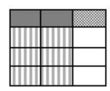
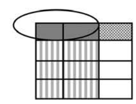
that, before I can get the brownies to school, my family has eaten 1/3 of them, so I have just 2/3 of a pan of brownies to take to my students. When I reach school, a former student stops by to say hello, sees the brownies, and asks if he can eat 1/4 of them. If I say yes, how much of the original pan of brownies would that student eat? In other words, what is 1/4 of 2/3? I show the pan of brownies, which is 2/3 full. I have cut a vertical line down the middle of the brownies in the pan, so students can clearly see that we have 2/3 of the brownies and that 1/3 of the brownies are missing. See Figure 11.15.

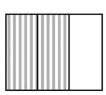
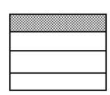
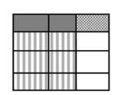
To link the concrete representation to its symbolic form, I will write the equation on the board. When we multiply fractions, the multiplication symbol is read as "of," so I will write the equation both ways:

$$\frac{1}{4}\text{ of }\frac{2}{3}=\square \quad \text{or} \quad \frac{1}{4}\times\frac{2}{3}=\square$$

(This is a good place to elicit students' ideas about how to proceed.) Because my friend has asked for 1/3 of the brownies in the pan, we agree that we need to divide the remaining brownies into four parts, as shown in Figure 11.15. I use horizontal cuts across the pan to divide the remaining brownies into 4 equal parts. We need ¼ of the brownies in the pan, which means we need one of the 4 parts. There are 2 brownie pieces in that part, which represents 2 out of the 12 brownies I baked. That is our answer: 1/4 of 2/3 = 2/12. Students who can calculate equivalent fractions may recognize that 2/12 is the same as 1/6, but

Figure 11.16 Modeling Multiplication with Fraction Squares and Overlays

Drawing Fraction Squares: 1/4 x 2/3		
Step 1	**Step 2**	**Step 3**
Show how much we have (the 2nd fraction). Draw a square. Draw vertical lines to show the total number of parts stated in the denominator, then shade the number of parts stated in the numerator.	Show how much we want (the 1st fraction). On the same square, draw horizontal lines to show the amount stated in the denominator, then shade the number of parts stated in the numerator.	The product is the are where the two models overlap.
 **2/3**	 **1/4**	 **2/12**

Fraction Overlays: 1/4 x 2/3		
Step 1	**Step 2**	**Step 3**
On one overlay, use vertical lines to show how much we have (stated in the 2nd fraction)	On a different overlay, use horizontal lines to show how much we want (stated in the 1st fraction)	Criss-cross the two overlays. The product is the area where the two models overlap.
 **2/3**	 **1/4**	 **2/12**

expressing the fraction in lowest terms is not necessary at this point and should not be allowed to distract students from the essential focus on multiplication.

The fraction squares and overlays we used to help students understand equivalent fractions provide a perfect two-dimensional representation of the brownie pan multiplication problem. To model 1/4 of 2/3, we first show the amount we have, which is stated in the *second* fraction in the equation (in this example, 2/3). See Figure 11.16. Students often become confused about which fraction to model. When we multiply fractions, the first factor always tells the portion we are taking. The second factor represents the original quantity, which is what we model. In the brownie example, we first modeled the 2/3 pan of brownies that we had at the beginning, and then we found 1/4 of that amount.

We can use fraction overlays to model the problem, or we can draw squares. If we are drawing squares to create our models, we show the original quantity using *vertical* lines to model the denominator (in this example, we need three sections), and then shade the number of pieces indicated by the numerator (in our problem, two). Then we consider the first fraction in the equation, which tells us how much of the model we will use. Our problem says to take 1/4 of that region, so we draw *horizontal* lines to divide the region into four equal parts, and then shade one of the four parts. The product is the region where the shaded slices overlap, which contains two squares. Those two brownie squares represent 2/12 of the original pan of brownies.

If we use fraction overlays to model this problem, the process is similar. First, we represent the second fraction (2/3) by arranging our square so that the lines run vertically. Then we use another overlay to represent the first fraction, 1/4. We crisscross the first fraction (1/4) on top of the second fraction (2/3) so that the lines on the top overlay run horizontally. On the overlays, the product is the region where the two shaded overlays overlap. Fraction multipliers are available from several publishers, including Learning Resources and EAI Education.

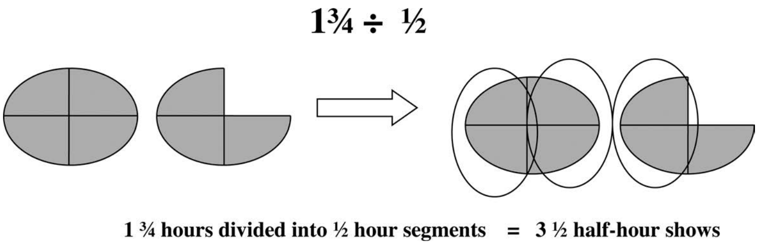
Using fraction squares or overlays to model multiplication is similar to the method described earlier for modeling equivalent fractions with overlays. When we created equivalent fractions, we were actually multiplying the given quantity by a fraction equivalent to one, such as 2/2 or 4/4. Normally, when we are using the whole unit, we would color in the entire square. However, that would make it difficult to see the original fraction when we crisscross the overlays on top of each other, so when we find equivalent fractions we just use lines and omit shading on the square that represents the whole. It may be helpful to demonstrate the process with a shaded-in template or drawing so students can connect the process for finding equivalent fractions to the multiplication algorithm, and then discuss why it is preferable to omit the shading when modeling equivalent fractions.

When students first begin to multiply a number by a fraction, they may be puzzled by the answer. In their previous experiences multiplying whole numbers, the product is usually bigger than the factors. (The exception is when we multiply by a factor of zero or one.) When we multiply by a fraction less than one, the product is always smaller than the original amount, because we are using less than a whole group. Concrete and pictorial representation helps students understand why this happens, and this understanding is essential in order for students to estimate products and judge whether an answer is reasonable.

Dividing Fractions

Division of fractions is one of the most challenging topics for students and their instructors, described by Liping Ma as "a topic at the summit of arithmetic" (Ma, 1999). In a much-discussed study, Ma asked teachers in the United States and China to represent and solve the following problem: 1¾ ÷ ½. In her samples, only 39 percent of American teachers could solve the problem, and only 4 percent of them could represent it. In contrast, 100 percent of Chinese teachers were able to correctly solve the problem, and 90 percent could also represent it. If American teachers struggle themselves with these operations, it is not surprising that their students will have difficulty as well. If we use the CPA sequence to introduce multiplication and division of fractions, we prevent much of this confusion, especially if we also provide a meaningful context for our problems. To solve Ma's problem of $1\frac{3}{4} \div \frac{1}{2}$, we first put it in a familiar context. Let us say that we have 1¾ hours of free time available to watch TV. How many 1/2-hour shows can we watch in 1¾ hours? Figure 11.17 shows a graphic illustration of this problem. In 1¾ hours, we can watch 3 complete 1/2-hour shows and 1/2 of another show. Note that, although we have 1/4 hour left over, that does not mean we can watch 1/4 of the last show. When we described how much free time we had, the hour was divided into quarter-hour segments. But when we are calculating the number of shows we can watch, a complete show lasts 1/2 hour, and so our unit changes to 1/2 hour. After we watch three complete shows, there will be 1/4 hour remaining, which is

Figure 11.17 Liping Ma's Problem



exactly half of a 1/2-hour show. Our answer therefore is that we can watch 3½ half-hour TV shows. Providing a realistic context for the problem makes it more accessible.

There are two approaches to teaching division of fractions. The method most commonly used in U.S. schools has been to teach students the inversion algorithm. For example, to solve the problem ¼ ÷ ⅔, we invert the second fraction, then multiply to obtain our answer: $\frac{1}{4} \div \frac{2}{3} = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$. In practice, this has sometimes meant telling students, "Don't ask why. Just invert and multiply!" The algorithm works, but explaining why is abstract, difficult, and involves skills students typically do not master until after they are expected to divide fractions. The alternative to teaching the invert-and-multiply algorithm is to teach students a division algorithm based on finding common denominators. This algorithm is not commonly used, but it has the advantage of being much easier to model and explain. To teach this algorithm, it is helpful to build on students' existing knowledge of division with whole numbers. For example, students might first solve the following problem that involves dividing whole numbers:

I want to make cookies. I have 8 cups of flour. My recipe calls for 3 cups of flour for each batch of cookies. How many batches of cookies can I make?

This is a measurement division problem. We have 8 cups of flour, and we need to divide them into groups of 3. The question is, "How many groups of 3 can we make from 8 cups?" The equation is written as follows: 8 ÷ 3 = ? Figure 11.18 shows 8 cups divided into groups of 3. We can make 2 groups of 3 cups, with 2 cups left over. These 2 cups provide 2 of the 3 cups, or ⅔ of the amount needed for the next batch of cookies. Therefore, from 8 cups of flour we can make exactly 2⅔ batches of cookies. The completed equation is 8 ÷ 3 = 2⅔. Once students can successfully solve this problem, they can apply their knowledge to the next problem, which is similar except that it involves dividing by a fraction rather than a whole number:

> I want to make cookies. I have 2 cups of sugar. My recipe calls for 3/4 cup of sugar for each batch of cookies. How many batches of cookies can I make?

Figure 11.18 Review: Modeling Division of Whole Numbers

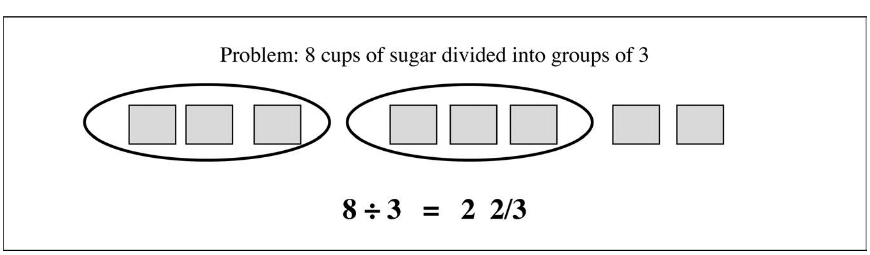
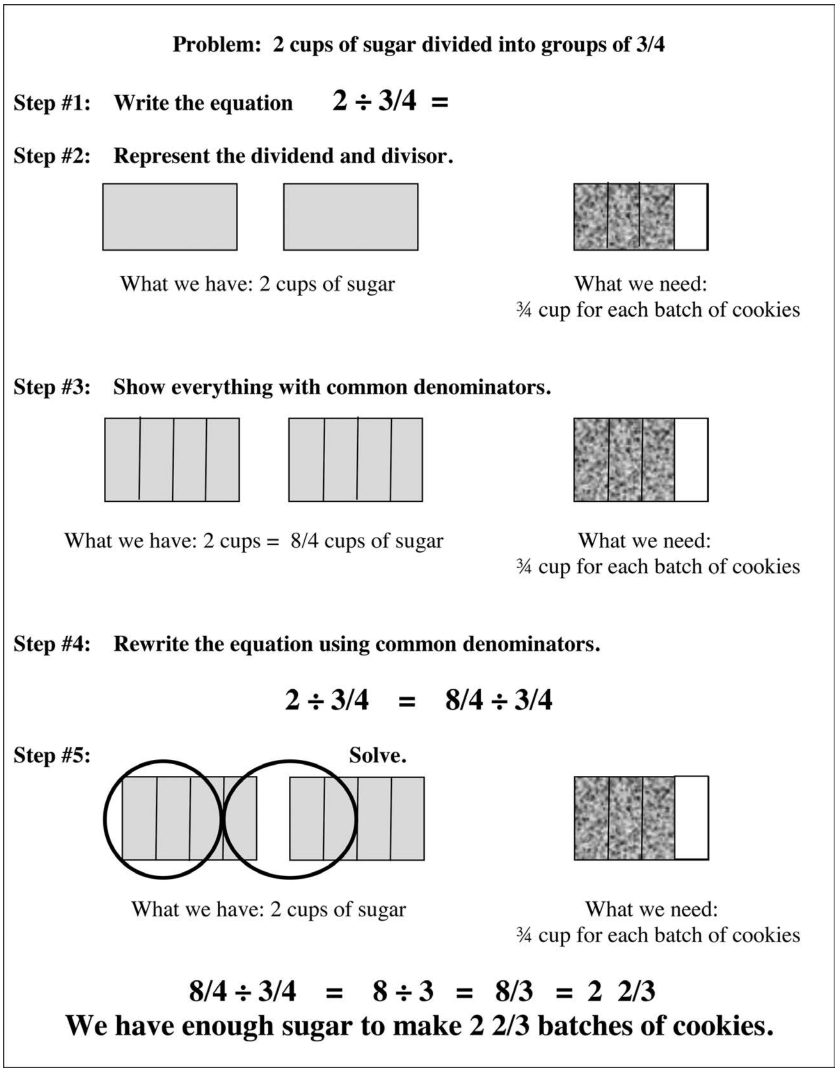


Figure 11.19 Modeling Division of Fractions



Like the previous example, this is a measurement division problem. We have 2 cups of sugar, and we need to divide them into groups of 3/4. The question is, "How many groups of 3/4 can we make out of 2 cups?" We write the equation as follows: $2 \div 3/4 = ?$

The skill steps provided in Figure 11.19 can help students solve the problem. First represent the dividend (2 cups of sugar) and the divisor (3/4 cup of sugar needed for each batch of cookies). Next, show both numbers using common denominators. In this example, one of the cups is divided into fourths, so we need to divide all the cups into fourths. When we do that, the two whole cups of sugar become $8/4$. The illustration of the 3/4 cup that is needed for each batch of cookies does not change. The equation, rewritten using common denominators, becomes $\frac{8}{4} \div \frac{3}{4} =$ ___. Both fractions refer to fourths of cups, but the denominator is actually irrelevant for solving the division problem. The question under consideration is *how many batches* we can make. We can find the answer by simply dividing the numerators without changing the denominators: $8 \div 3 = 2\frac{2}{3}$. If we have 2 cups of flour, we can make $2\frac{2}{3}$ batches of cookies. Note that this is the same equation used in our previous example when we divided whole numbers. By connecting the division of fractions to students' previous understanding of whole numbers, we build more meaningful understanding. Although the division algorithm based on finding common denominators is seldom taught, it is a much easier algorithm to model and explain meaningfully. Van de Walle (2019) suggests using this procedure, and we believe it can build greater understanding for those students who have struggled with more traditional approaches.

Just as students may have been confused by the answers they obtained when multiplying fractions, the results of dividing fractions may also surprise them. When we divide whole numbers, the quotient is smaller than the original amount (the dividend). The use of concrete and pictorial models allows students to see that, when we divide the unit into fractional pieces, we are creating groups that are less than one. It takes more small groups to equal the whole, so our quotient will be larger than our original number. Providing time for students to explain and justify their solutions will help solidify this concept.

Although NCTM's standards advocate using representation to develop students' understanding in all areas of mathematics, examination of popular textbooks suggests that instruction in advanced fraction concepts relies primarily on abstract words and symbols. Interventionists will therefore need to add visual representation to many of the commercially available materials used to teach higher-level fractions. See the online resources for a list of materials and videos for teaching fractions.

Decimals

Most students have some previous experience with decimals because our monetary system uses decimal values. Some students may also have experience with baseball statistics, which use decimals to report batting averages and other player information, or other sports statistics. Since decimals use the base-ten number system to express fractional quantities, we can facilitate students' ability to understand decimals by connecting decimal instruction to their previous experiences with fractions and place value.

The same manipulatives used to introduce fractions can also be used to represent decimal values. Often teachers use a place-value chart to introduce decimal values, but a place-value chart is a form of two-dimensional representation. To follow the CPA continuum, interventionists should first introduce concrete examples of decimals. Decimals are just another way of writing fractions that have denominators of 10, 100, 1000, and so forth, but students may fail to make this connection without concrete representation. Using fraction circles or bars that show tenths helps connect decimals to students' prior understanding. DigiBlocks, which were described in Chapter 7, include blocks specifically designed to represent tenths. Because these miniature blocks are one-tenth the size of the DigiBlocks unit block, they provide a concrete model that shows the relative size of a tenth compared to a whole unit. In Figure 11.20, a student is modeling the number 15.2 with DigiBlocks.

Students can also use the fraction towers that were described earlier in this chapter to create models of decimal tenths. In addition to the standard sets of fraction bars, versions are available that are labeled on one side as a fraction and on the other side as the decimal equivalent, so students have a concrete model that illustrates how the same amount can be represented as either a fraction or a decimal. See Figure 11.21. In addition, measurement models such as meter sticks and number lines can be used to help students understand decimal values.

Base-ten blocks are another excellent tool that can provide concrete representation of decimal values and also help students connect decimal notation to their previous experiences

Figure 11.20 Modeling Decimals with DigiBlocks

Modeling 15.2 with DigiBlocks

Figure 11.21 The Relationship Between Fractions & Decimals

with place value. When we use base-ten blocks to represent whole numbers, a unit cube represents 1.0, a rod is worth 10, a flat is 100, and the big cubes represent 1000. We can use the same cubes to represent decimal numbers by changing which cube we designate as having a value of 1.0, as shown in Figure 11.22. Students must understand that a decimal point is a symbol used to show the location of the ones place, so they can therefore use the decimal point to identify the place value of all the digits in a number. Base-ten blocks allow students to create concrete models of decimal numbers, which makes the abstract concept of place value more comprehensible.

Base-ten blocks, Digi-blocks, fraction bars, and fraction circles are all examples of ways to represent decimals concretely. When students are ready to transition from concrete representation to pictorial representation, graph paper allows students to create two-dimensional models that are similar to the concrete models they formed with base-ten blocks.

When students mark off a 10×10 section of paper, the shape is similar to the 10×10 flat. If this square is labeled as the unit, or 1.0, then a column of 10 squares would represent 0.1 and each individual square would have a value of 0.01. See Figure 11.23. Students can use graph paper drawings to demonstrate their understanding of decimal values and to make comparisons between decimals. For example, students working with abstract numbers sometimes focus on the face value of the digits in a decimal and forget to consider the digit's place value. This can lead them to mistakenly conclude that a number like 0.4 is less than 0.18 because 4 is less than 18. When students use concrete or visual models to represent the quantities, relative values are more easily perceived. Figure 11.24 provides an example of this comparison.

Figure 11.22 Representing Decimals with Base Ten Blocks

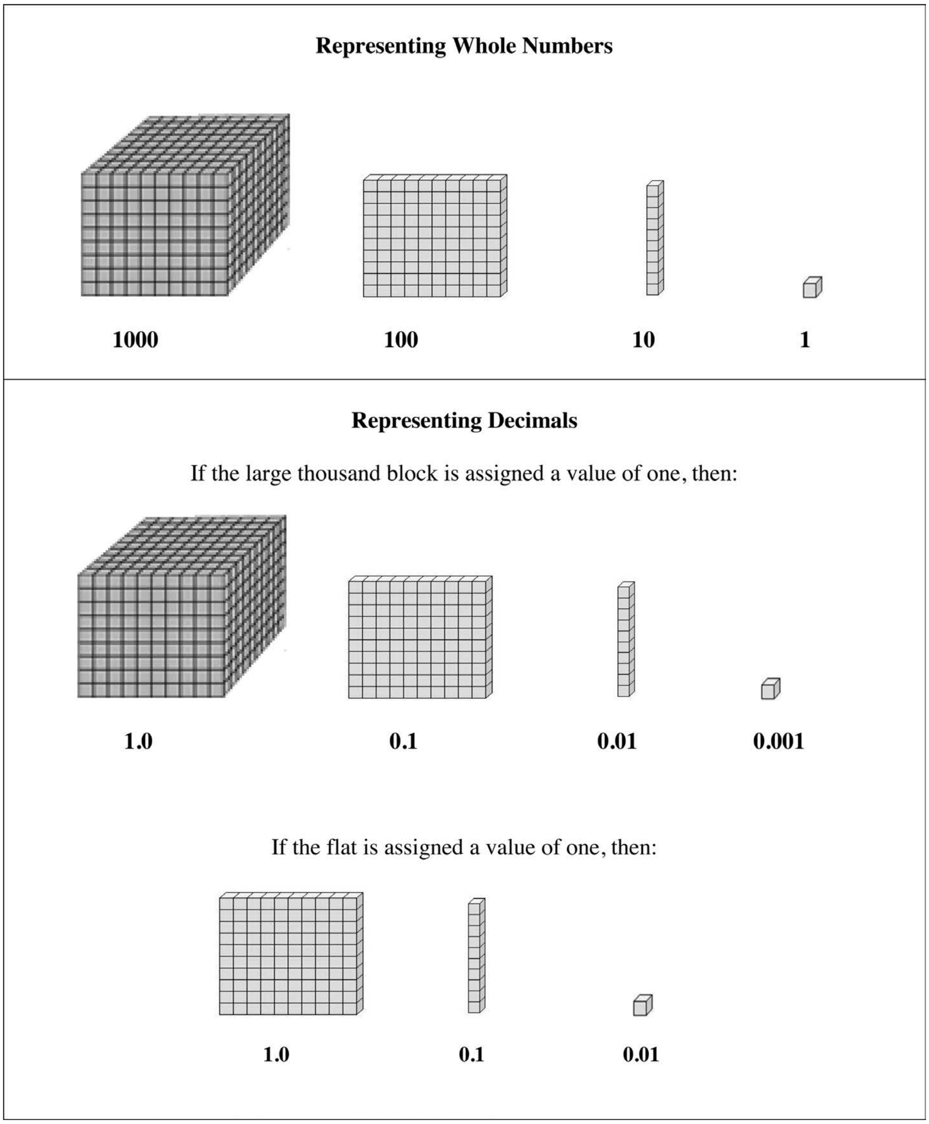


Figure 11.23 Representing Decimals on Graph Paper

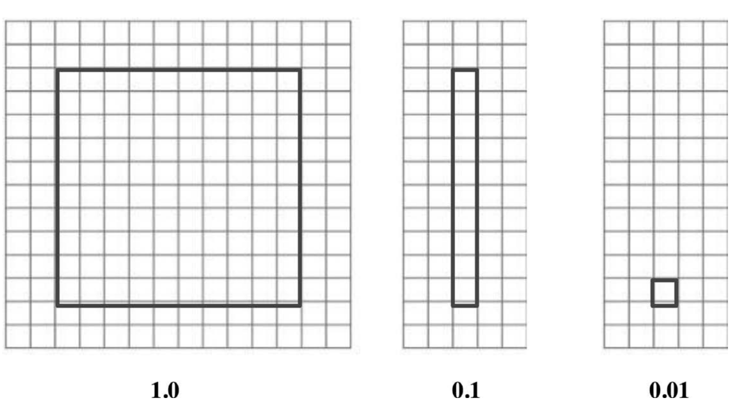
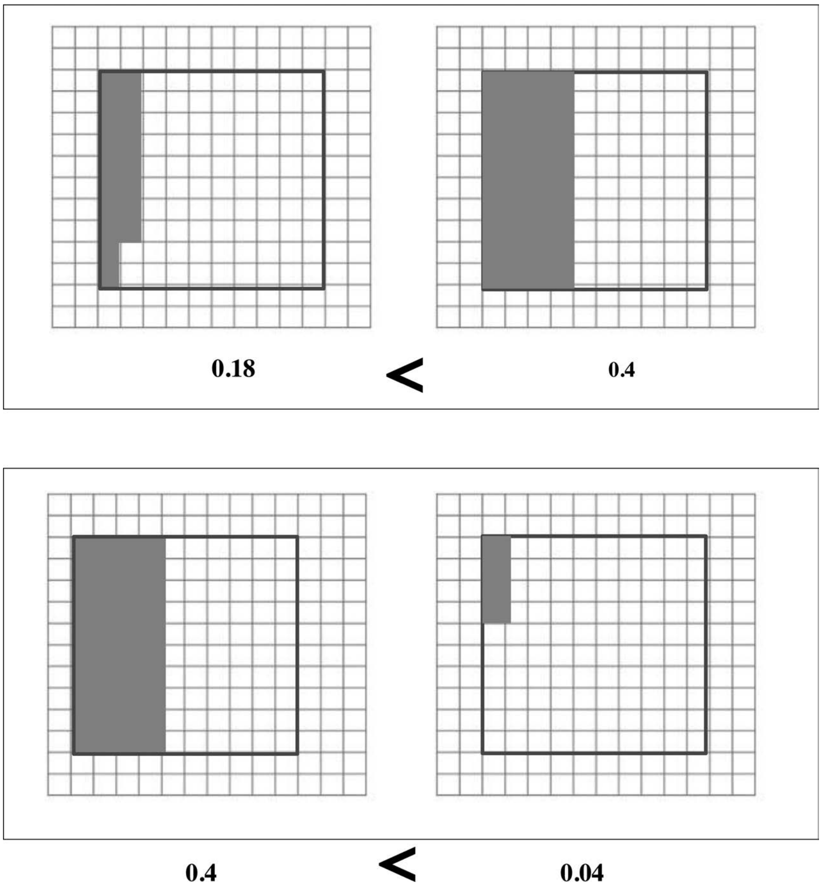


Figure 11.24 Comparing Decimals



A similar model can help students understand decimal equivalence. When we add zeroes to the right of a decimal, its value is unchanged. For example, 0.1 is equivalent to 0.10 and 0.100. Using graph paper, students can illustrate each of these decimal numbers and see that they cover the same area. See Figure 11.25. The concept of decimal equivalence can also be modeled with number lines, as shown in Figure 11.25.

When students begin to perform operations with decimals, graph paper models continue to be useful. One of the most frequent errors students make with decimal computation is to ignore place value. For example, when asked to add quantities like 2 and .8, they may forget to line up the decimal points and so report the sum as 10 instead of 2.8. Modeling the problem with blocks or graph paper can clear up the confusion. See Figure 11.26. Again, concrete or visual representations will clarify the problem and help students avoid such errors in the future.

To represent multiplication of decimals, students can use the same models they used when representing multiplication of fractions. For example, to multiply 0.1×0.3, we can use the same fraction overlays used to multiply fractions. We model 0.3 and 0.1, as shown in Figure 11.27. When we crisscross the overlays, the region where the two overlap is the product, 0.03. Asking students to think about and explain why this answer is expressed in hundredths can help them solidify decimal concepts.

When we multiply a quantity by a fraction, the resulting product is smaller than the original factor. The same is true when we multiply by a decimal. For example, if we consume

Figure 11.25 Modeling Decimal Equivalence

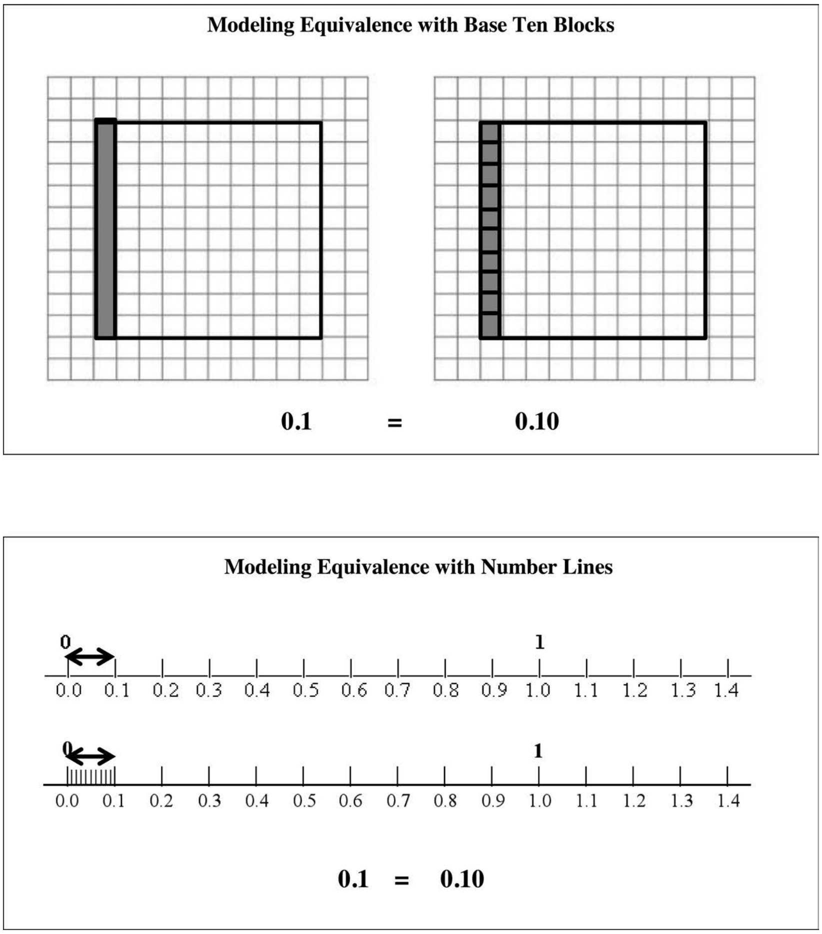
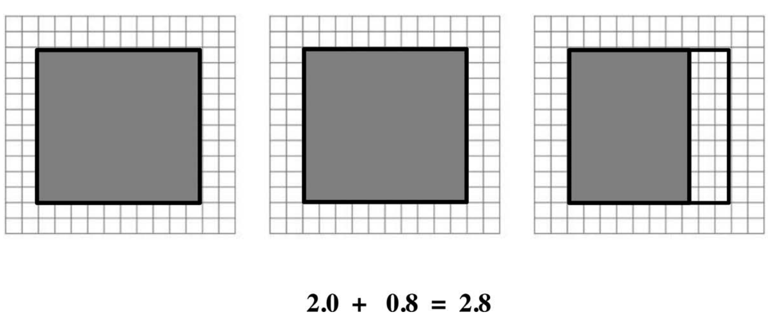


Figure 11.26 Adding Decimals



1/2 of 4 cookies, we have eaten 2 cookies. If we express the factor 1/2 as a decimal, we will obtain the same result: $.5 \times 4 = 2$. In contrast, when we multiply by whole numbers greater than one, the product is always larger than our original amount. Physical and pictorial models like those provided in Figure 11.27 can help students understand these principles; such understanding is essential in order for students to learn to estimate answers and judge whether an answer makes sense.

Figure 11.27 Modeling Multiplication with Overlays

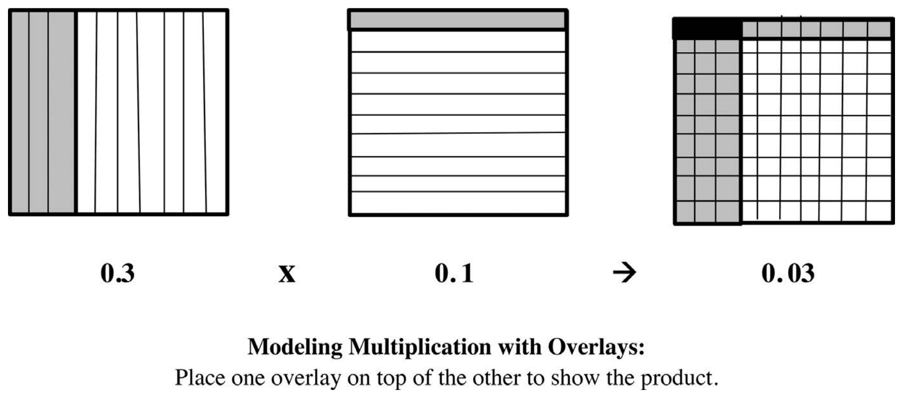
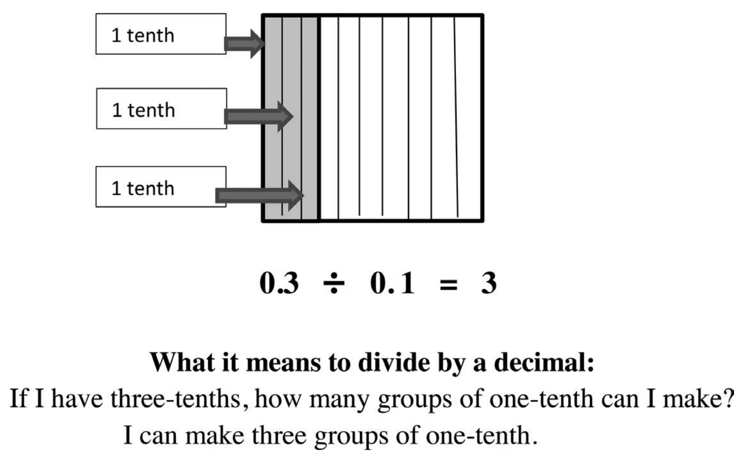


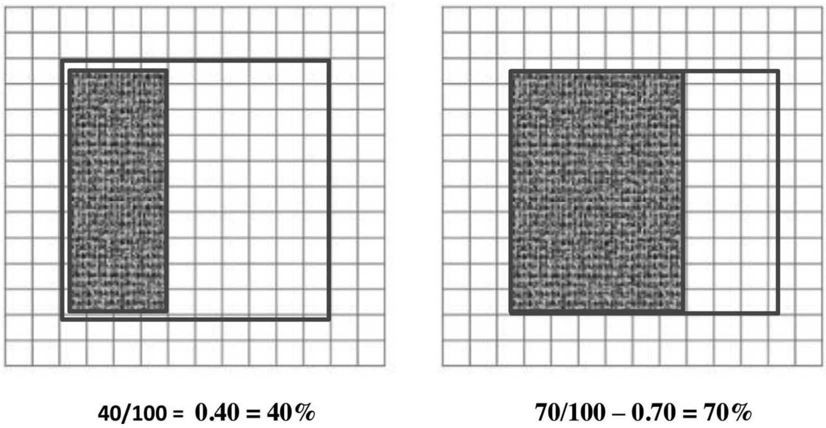
Figure 11.28 Modeling Dividing by a Decimal



When we divide by a whole number, the quotient is smaller than the original quantity. When we divide by a fraction or decimal, the resulting quotient is greater than the original amount, because we have divided the original quantity into many small pieces. Students may initially find these results puzzling because they are used to dividing by whole numbers. Creating a model helps students understand the effects of dividing by a decimal, as illustrated in Figure 11.28.

Decimals are typically introduced in fourth grade. Although there is strong research support for using manipulatives and pictures to introduce any new concept or procedure, operations with decimals are too often introduced as rote procedures. Although students may learn to execute the computations accurately, they will have trouble applying their knowledge in problem-solving situations if they do not understand the underlying principles. Because upper elementary and middle school math materials currently rely on abstract representation for most lessons (Alkhateeb, 2019; Gersten et al., 2009; van Garderen, Scheuermann, Poch, & Murray, 2018), interventionists will frequently need to add concrete and visual representation activities to support students who receive tiered interventions.

Figure 11.29 Fraction-Decimal-Percent Equivalents



Percent

Percentages are commonly encountered in everyday situations. Meteorologists report the chance of storms as a percentage: "There is a 40 percent chance of rain today." Sales tax and income tax are calculated based on a percentage of cost or income. The term "percent" derives from the Latin meaning "per hundred," and percent provides another way to represent fractional or decimal hundredths. We can therefore use the same models we introduced for fractions and decimals to represent percent. Activities that explicitly link fraction-decimal-percent equivalents can help students construct an understanding of percent. For example, we can shade a portion of a 100-square section of graph paper and express the quantity as a fraction, a decimal, and a percent, as shown in Figure 11.29.

Our representations should include quantities greater than one as well as models of quantities less than one. These larger numbers, such as 150% or 200%, are most easily understood when modeled in relation to an illustration of 100%. See Figure 11.30.

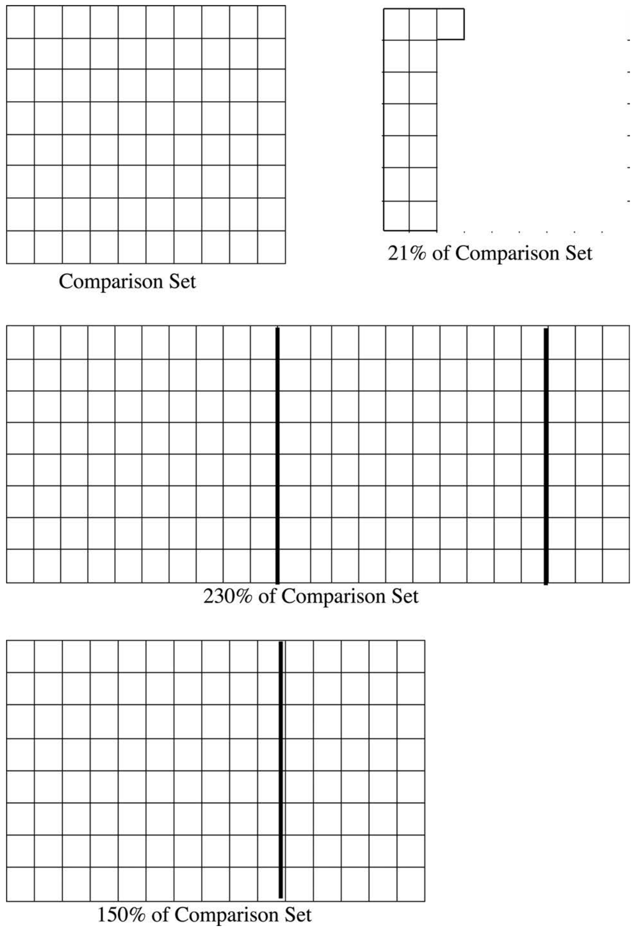
Using a variety of types of representations to model a single concept deepens students' conceptual understanding. Therefore, we should not limit our representations of percent to using 100-square grids, but also include models that use pattern blocks, geoboards, meter sticks, number lines, and other concrete and visual images, just as we did when introducing fractions and decimals.

Intensifying Instruction

Intensifying Instruction During Interventions

Although the process for teaching rational numbers to students who receive tiered supports is similar to instructional strategies presented in core (Tier 1) instruction, there are important differences. Many educators who provide math interventions do not have access to a validated program where intensive intervention practices are already built into the program. Others work with students who require even more individualized supports. Ideas for intensifying instruction to meet the needs of learners receiving tiered support were discussed throughout this chapter. Here is a summary of some of the many ways to intensify instruction during interventions.

Figure 11.30 Using Comparison Sets to Model Percent



1. Interventionists who have access to a program that is validated for use with students receiving Tier 2 supports should implement the program with fidelity. If a validated program is not available, then intensify the instruction provided in the textbook or other resources that are available by adding or increasing use of the strategies described below. It is not enough to simply follow the program provided during core instruction, because that has already been shown to be ineffective for the student. Instead, add additional evidence-based supports to intensify instruction. Students who continue to struggle after receiving Tier 2 support, then that student needs even more intensive support during Tier 3 interventions. If a validated program was used during Tier 2 instruction, intensify it further by increasing use of the supports described below. If a validated program was not used for Tier 2 instruction, increase the intensity of instruction the same way, by increasing use of the supports described below.
2. Use systematic instruction. Select objectives carefully. Sequence them from easiest to hardest, and make sure that pre-requisite skills are mastered before introducing more complex content. If students struggle, objectives can be further broken down into component parts or steps. If a student struggles to complete all the steps in a single lesson, then the lesson could be broken down to focus on only one or two steps each day. Although it will take longer to introduce the complete procedure, this approach often saves time in the long run because it reduces the need for reteaching. To avoid

overwhelming students' cognitive capacity, pace instruction so that students solidify their understanding of one concept or skill before introducing another.

3. Use explicit instruction. Follow the guidelines described in Chapter 5. If the available materials do not use this high-leverage practice, then modify the lesson to include all the elements of explicit instruction.
4. Give students a written list of steps to follow, and teach them to refer to the list as they work. Many students who struggle with mathematics have deficits in executive functioning. Teaching them to monitor their progress by checking off steps has been shown to increase achievement.
5. Follow the CPA continuum. Always begin at the concrete level, and allow students sufficient time exploring and mastering math concepts with manipulatives and pictorial representation before expecting them to solve problems using only abstract words and numbers. Explicitly connect the concrete and pictorial representations to the abstract algorithm to build deep understanding. When students can explain the meaning of each step, then they are ready for interventionists to fade the concrete and visual supports and focus on developing procedural fluency with abstract representation.
6. Use precise academic language when you model mathematical procedures. Emphasize vocabulary in each lesson, and have students practice using the academic vocabulary themselves. Supplementing verbal language with gestures has also been shown to increase understanding and retention for some students.
7. Have students explain what they are doing, and why they are doing it this way. Asking students to explain their reasoning helps them solidify understanding, and also provides valuable formative assessment information that can be used to refine instruction. Core curriculum materials increasingly stress the importance of communication in mathematics. Too often students receiving interventions have learned to use tricks and follow steps by rote, without developing conceptual understanding. Asking students to explain their own reasoning, and to understand and critique the reasoning of others, is important to develop mathematical proficiency.

Summary

Rational numbers involve some of the most challenging content students encounter, yet most current instructional materials provide limited opportunities for students to experience concrete or pictorial representations of these difficult concepts. Visual representations allow students to organize information, describe mathematical relationships, and communicate mathematical ideas to others. The process of representing their ideas helps students construct meaning, as well as organize and clarify their thinking. Research indicates that understanding follows a developmental sequence, beginning at the concrete level when students physically manipulate concrete objects, then progressing to pictorial representations such as drawings, tallies, and graphs, and finally moving to abstract words and symbols. Linking various representations of the same mathematical concept or procedure deepens students' mathematical understanding. Providing explicit strategies and rigorously following the CPA sequence will allow all students to become proficient with rational numbers.

12
Problem-Solving

The goal of mathematics instruction is for students to be able to solve real-world problems. Mathematicians define problem-solving as "engaging in a task for which the solution method is not known in advance" (NCTM 2000, p. 52), or "finding a way out of a difficulty, a way around an obstacle, attaining an aim which is not immediately attainable" (Polya, 1965). Both the Principles and Standards for Mathematics (NCTM 2000) and the Common Core State Standards (National Governors Association Center for Best Practices, 2010) emphasize problem-solving at every grade level. However, international assessments show that American students struggle when asked to solve problems, and the task is especially difficult for students with deficits in mathematics (Geary, 2003; Pfannenstiel et al., 2015). The metacognitive competencies required for problem-solving are precisely those skills that students with mathematical disabilities find difficult, including: processing the language of the problem and understanding what is being asked, identifying and organizing relevant information, selecting a problem-solving strategy, remembering and executing the strategy steps in the proper sequence, performing necessary computations and accurately recording solutions, and checking to make sure the computation was executed successfully and that the answer makes sense. For students who have difficulty with executive functioning, problem-solving is a daunting task.

Problem-Solving in the Core Curriculum (Tier 1)

Math programs designed for use in the core curriculum use a variety of different approaches to teach problem-solving. Many use a variation of Polya's (1945) four-step process, which includes: (1) understand the problem; (2) devise a plan; (3) carry out the plan; (4) look back and reflect. These are critical steps for successful problem-solving, but they do not provide enough guidance for many students. Students with language processing problems, reading problems, or for whom English is a second language may not understand the problem. Students with deficits in executive functioning will have difficulty devising a plan or executing a plan, and often struggle when asked to reflect on what they have done. They need more detailed support at each step than what is provided in most core programs. In addition, core materials generally encourage students to select from a menu of problem-solving strategies, including representing the problem by drawing a picture, acting it out, using a model, or making a table or chart (Van de Walle, 2019). All of these are useful strategies, but individuals with deficits in metacognitive functioning will be easily overwhelmed by the choices. Instead of evaluating the problem and selecting the most appropriate strategy, they often resort to a simple "guess and check" approach. They may look for superficial clues such as keywords, or assume that every problem presented can be solved using whatever operation has just been taught (Carpenter et al., 1999).

Teaching students to identify underlying structures, also called story schema, and then use those structures to solve word problems, is an evidence-based strategy recommended for use with all students. The IES Practice Guide for improving problem-solving in general education classrooms advocates emphasizing underlying story schema in the core curriculum (Woodward, Beckman, Driscoll, Franke, Herzig, Jitendra, Keodingel, & Ogbuerti, 2012). The Association of Mathematics Teacher Educators (AMTE) also emphasizes the importance of helping students to identify and use story structures in their *Standards for Preparing Teachers of Mathematics*:

> Well-prepared beginning teachers focus on sense-making and reasoning when they prepare students to grasp the full meaning of a problem by comprehending the entire situation and trying to use structures, such as schema, properties of the operations, and representations to come to a reasoned solution. (AMTE, 2017, p. 83)

What does it mean to focus on structures or story schema? The underlying structure is the framework of the problem. For example, if we were teaching students to build a house, we would want them to recognize basic structural components such as the foundation, the walls, and the roof. Although teachers may not be familiar with the term "underlying structures," educators in many disciplines routinely use the concept. Reading materials contain different underlying structures, and we routinely teach students to identify components such as characters, setting, and plot found in narrative materials, or to identify the topic sentence, supporting details, and conclusion in a persuasive essay. We teach students to identify the elements of a scientific experiment, such as the independent variable, the dependent variable, and the control. We teach underlying structures to help students understand the content. In mathematics, as in other disciplines, specific structures always occur in particular problem types. For example, in a part/whole problem there are always two or more parts and a whole. We can subtract to separate the whole into its parts, or add the parts together to form the whole. In a comparison problem, we compare a larger item to a smaller item to find the difference. The term "underlying structures" describes these common features found in every example of a given problem type. Teaching students to recognize underlying structures, or schema, helps them organize information found in a word problem. When they arrange the important information on a graphic organizer, it reduces cognitive load and makes the solution strategy obvious, which facilitates effective problem-solving.

Concepts such as *part/whole* and *compare* are used in core materials, but these materials seldom highlight the underlying structures in each word problem they present. Instead, textbooks often focus on a different strategy in each lesson, so students may not see how underlying structures apply in every word problem they encounter. In addition, core materials may teach students to identify the structures, but they seldom teach students how to use underlying structures to solve problems. While some students perform well despite this omission, many students struggle, as evidenced by American students' poor performance on national and international achievement tests.

Figure 12.1 Ineffective Strategies

<u>Avoid</u> These Ineffective Strategies!

KEY Words	CUBES
Words that signal addition: *more, altogether, in all, total* Words that signal subtraction: *left, difference, fewer, remain* Words that signal Multiplication: *product, times, each* Words that signal division: *split, half, share equally*	C- Circle the number. U- Underline the question. B- Box the key words. E- Eliminate what you don't need. S- Does your answer make Sense?.

These strategies do not work consistently.
Even more problematic, they do not teach the underlying structures that students must understand to succeed in higher level math.
Do not teach students to rely on these strategies.

An additional concern with core instruction is that studies have found that ineffective problem-solving strategies are taught in many classrooms (Karp, Bush, and Dougherty, 2014; Powell & Fuchs, 2018). To support their students, teachers supplement the primary textbook with resources obtained from internet sources such as Google.com or Pinterest.com. (Karp, Bush, and Dougherty, 2014; Opfer et al., 2016), but unfortunately, ineffective strategies abound in these online resources. One example of a popular but ineffective strategy involves teaching students to use keywords, or the CUBES strategy (See Figure 12.1).

With this strategy, students are taught a list of words that cue a particular operation. For example, students learn that the word "more" suggests addition, as when you have one cookie and someone gives you more cookies, so you add to find the total. The keyword strategy is ineffective, however, because the same word can also be used in a subtraction problem, as when one person has "more" cookies than another and you need to find the difference. Students who rely on keywords will routinely make mistakes if they have not learned to read and understand the whole problem. In addition, although keywords appear in many of the word problems presented in textbooks in the lower grades, they are less prevalent in higher-level word problems, and they are ineffective when solving the multi-step problems that dominate upper elementary practice. Therefore, while students who rely on keywords and the CUBES strategy may successfully solve word problems in the early grades, their performance plummets when they encounter word problems in higher grades. Even more problematic, studies show that students who rely on keywords frequently jump to computation without fully understanding the problem (Drake and Barlow, 2008; Heng & Sudarshan, 2013). When students have learned to rely on keywords, rather than learning to understand the problem and identify underlying story schema, they struggle with higher math content and real-life applications. The Standards for Mathematical Practice stipulate that students should "make sense of problems" (National Governors Association Center for Best Practices, 2010), but relying on keywords represents a shortcut that actually interferes with a student's ability to make sense of the problem. The *Standards for Preparing Teachers of Mathematics* (2017) specify that well-prepared teachers "understand that short cuts such as searching for keywords are not effective … "(AMTE, 2017, p. 82). For all these reasons, researchers and mathematics educators urge teachers to stop teaching the keyword or CUBES strategy (AMTE, 2017; Karp, Bush, & Dougherty, 2014, 2019; Van de Walle, Karp, and Bay-Williams 2019). Instead, teach students to use underlying structures to solve word problems.

Core instruction meets the needs of many students, but national and international assessments show that many other American students struggle. Fourty percent of our students scored below the NAEP Basic level on the tests administered in spring 2019, and so may require tiered support available through RtI.

Teaching Problem-Solving During Interventions (Tier 2 & Tier 3)

Students who have struggled with mathematics benefit when they are explicitly taught how to recognize the underlying structures found in word problems and use those structures to solve the problems (Jitendra et al., 2015; Pfannenstiel et al., 2015; Powell & Fuchs, 2018). Explicit strategy instruction on common underlying structures is considered an evidence-based practice during mathematics interventions (Jitendra et al., 2015). The IES Practice Guide for *RTI in Mathematics* recommends:

> Interventions should include instruction on solving word problems that is based on common underlying structures … When students are taught the underlying structure of a word problem, they not only have greater success in problem-solving but can also gain insight into the deeper mathematical ideas in word problems. (Gersten et al., 2009, p. 26)

When core materials talk about story structures, the seldom teach students how to use those structures solve the problems, so interventionists will need to use a program that has been validated as effective for use with students who require Tier 2 support. As explained in previous chapters, a validated program is defined as a program where "there is positive evidence, collected during at least one well-conducted randomized control trial, that the program improves the mathematics outcomes of students with mathematics disabilities in a Tier 2 intervention" (Powell & Fuchs, 2015, p. 183). Several validated programs have been developed to help students who struggle with mathematics learn to use underlying structures, or schema, to solve word problems. Although they may use slightly different language, different ways of modeling the relationships among the numbers, and different heuristics to guide student thinking, validated programs all use explicit instruction to teach students to identify underlying structures in word problems and then to use the structure to solve the problems. The Academic Intervention Tools Chart from the National Center on Intensive Intervention, available online at www.intensiveintervention.org, provides summaries of efficacy studies of mathematics intervention programs to assist educators in finding effective intervention materials.

If a validated program is not available, then interventionists will need to provide extensive adaptations to intensify instruction to meet the needs of students who require Tier 2 support. Students who continue to struggle after receiving Tier 2 support can receive more individualized support during the intensive interventions provided at Tier 3. Tier 3 interventions may be built on a validated program, if one has been used during Tier 2, but if a validated program has not been used in Tier 2, then teachers must build on the existing structures (McInerney, Zumeta, Gandhi, & Gersten, 2014). In the rest of this chapter, we provide a description of how to intensify instruction during interventions to help students master mathematical problem-solving.

Adding and Subtracting Part/Whole Problems

Mathematics textbooks prevalent in the U.S. organize word problems in different ways, but most focus on two underlying structures in addition and subtraction word problems: *part/whole* and *compare*. Part/whole problems always contain a whole, or total, that is separated into two or more parts. You add the parts together to form the whole, or subtract a part from the whole to find the part that remains. Mathematicians often separate the *part/whole* problems into two types: *group problems (*sometimes labeled *total*), and *change* problems, and recommend different strategies for solving the two types of problems. However, because most current math textbooks do not differentiate between *change* and *group* problems, but instead simply focus on part/whole relationships, we will briefly describe how group and change problems differ, and then switch to the term *part/whole* problems as we discuss instructional strategies that support student understanding of both types of problems.

Group or *Part/Whole Problem*s involve parts that are combined to make a whole. A *group* problem might ask about the total number of students in the classroom, some of which are boys and some of which are girls. Together, the two parts (boys and girls) form the whole (children in the classroom). An example of a *group* or *part/whole* problem would be:

> Allison had six red M&M's and five yellow M&M's. How many M&M's did she have in all?

The whole, or superordinate set, is the total number of M&M's. The parts, or subordinate sets, are the red and yellow M&M's. There is no change over time; the problem asks about the quantity of parts and the whole at one particular moment. Here is another example of a *group* problem:

> For Mother's Day, Shameka picked a bouquet of tulips, hyacinths and daffodils for her mother. She picked 12 flowers. If five of the flowers were daffodils and three were hyacinths, how many were tulips?

This is an example of a *group* type of part/whole problem because it describes parts (tulips, hyacinths, and daffodils) that combine to make a whole (flowers). The story describes the quantity of flowers at a given moment, not change over time. The critical concept for students to understand is that the whole is equal to the sum of its parts. Group problems always contain a whole that is separated into two or more parts.

Change problems describe a scenario where the quantity of an item changes over time. For example, the problem might involve the number of children in the classroom. At the beginning of the story, there is a given number of children in the room. Then some more children arrive or some children leave, so the number of children at the end of the story is different from the number in the room at the beginning of the story. In *change* problems, the story is always about the same type of item, but the quantity of that item changes over time because some items are added or subtracted. Here are two versions of a *change* problem:

> Melissa had three cookies. Her friend gave her two more cookies. How many cookies does she have now?
>
> Melissa had five cookies. She ate two of them. How any cookies does she have now?

Both stories are about Melissa's cookies. In the first example, both parts are given (i.e., the cookies Melissa had at the beginning, and the cookies that her friend gives her), and the question asks the student to find the whole. In the second example, the whole is the number of cookies Melissa had at the beginning, and the parts are the cookies she ate and the cookies she had left. When the whole is unknown, we add the parts together to find the missing information. When a part is unknown, we can find the missing part by subtracting the known part from the whole. *Change* problems can involve addition or subtraction and the information can be presented in any order, but the problems always describe a situation where the quantity of the same item changes over time.

Core materials often do not differentiate between *group* and *change* problems. Instead, they treat both types of problems as variations of *part/whole* problems. In an addition *change* problem, the quantity present at the beginning of the problem represents one part, and the quantity added to it is another part. The parts combine to form the whole or total. In a *change* problem that involves subtraction, the quantity present at the beginning of the problem is the whole or total, and then one part is subtracted, and the remainder is the other part. Interventionists who do not have access to a validated program build instruction on existing materials, such as the textbook used in the student's general education classroom. Since most basal textbooks do not distinguish between *group* and *change* problems, for the rest of the chapter we will simply use the term *part/whole* problems.

Emphasize Vocabulary

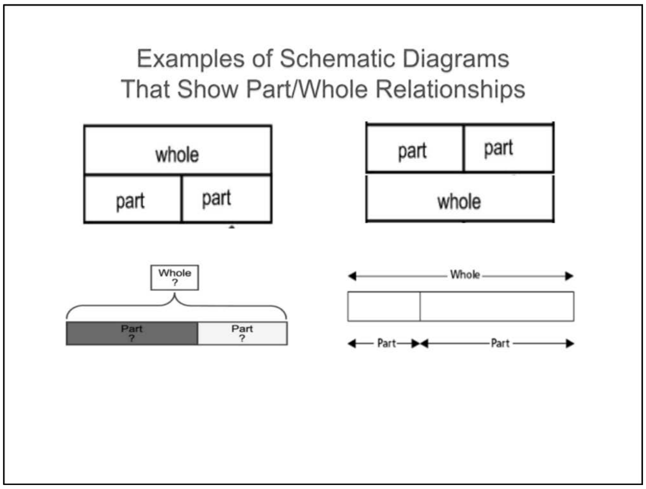
There are many ways to intensify problem-solving instruction during interventions. Stressing vocabulary words like *part* and *whole* during every lesson is a simple adaptation that can be very effective. The hand gestures we introduced in Chapter 8 are another powerful way to reinforce underlying story structures. To illustrate addition in *part/whole* problems, put the members of one part in your left hand and extend it in front of you as you say "part." Then put the members of the other part in your right hand and extend it in front of you as you say "part." Finally, bring your two hands together, cupping the combined parts together as you say, "whole." Initially this is done with actual objects in your hands so the students can see a concrete example of how the parts combine to form the whole. Eventually, the same gestures can be used without actual objects. Teach students to use the hand gestures and say "part, part, whole" when they are adding parts to form the whole. To model subtraction, reverse the process. Begin by cupping your hands together in front of you and saying "whole," and then remove one part as you say, "part," and extend your left hand forward. Finally, extend your right hand forwards as you say, "part that's left" or "remainder." Emphasizing underlying structures through gestures and explicit vocabulary instruction is a simple but effective way to intensify instruction.

Creating Schematic Diagrams of Part/Whole Problems

Schematic diagrams are an evidence-based method of modeling word problems during interventions (Jitendra et al., 2015). A variety of diagrams are available that show underlying structures. Figure 12.2 shows examples of some of the different diagrams commonly found in textbooks. They may be called bar models, tape diagrams, strip diagrams, or other names, but all are graphic organizers that highlight the underlying structures in mathematical problems.

Arranging the information on a schematic diagram/graphic organizer is helpful because the student no longer needs to hold the facts and their relationships in working memory. This reduces the cognitive load and allows the student to focus on analyzing and solving the problem. In addition, since graphic organizers involve non-linguistic representation,

Figure 12.2 Examples of Schematic Diagrams



students with language processing problems or for whom English is a second language benefit because the amount of verbal explanation can be minimized.

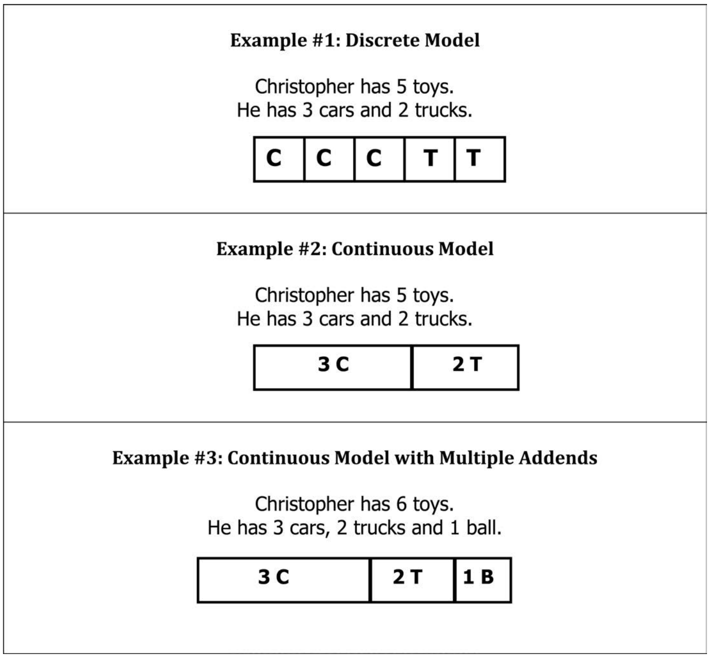
In most core programs, students are exposed to a variety of two-dimensional representations, and are also encouraged to create their own representation to ensure that their drawing is personally meaningful. Students who struggle mathematically experience great difficulty when they attempt to design their own graphic representations, and research studies have found that the representations they create are often of poor quality or focus on superficial features that do not reflect the problem's underlying structure (Montague & Jitendra, 2006; van Garderen, Scheuermann, & Poch, (2014). Therefore, we do not recommend having students create their own representations during tiered interventions. Instead, provide explicit instruction on how to use a specific type of diagram for each problem type. The schematic diagrams help students organize the information from any addition or subtraction problem into a recognizable pattern that facilitates problem solution. This vastly simplifies the cognitive task.

Students' first experiences with model drawing should use very simple illustrations. When students first transition from concrete representation to the use of more abstract drawings, they need to use a discrete method of modeling, which means they draw one small square to represent each object in a problem. For example, the following problem might be presented:

Christopher has five toys. He has three cars and two trucks.

Students functioning at the concrete level would need to model the problem with actual objects. When they progress to pictorial representation, they would initially draw three squares to represent the cars and two squares to represent the trucks to help them understand the relationship between concrete and pictorial representation. They can place different colored counters in the boxes to represent three cars and two trucks. As they gain proficiency, they would progress to drawing pictures to represent the cars and trucks, and to labeling each square with a word ("cars" or "trucks") or an abbreviation such as "C" for car and "T" for truck. See the first example in Figure 12.3. Once a student masters the concept of cardinality, he can progress from a discrete method where each object is represented by a single block to a continuous model where unit bars represent multiple objects. For the example above, a student using a continuous model would draw a unit bar to represent the total

Figure 12.3 Discrete and Continuous Model Drawings



number of toys and label the bar "5 toys." Then he would divide the bar into two sections with a vertical line, and label one section "3 cars" or "3 C" and the other "2 trucks" or "2 T." Example #2 in Figure 12.3 shows a continuous model. This same type of model can also be used to represent problems with multiple addends, as shown in Example #3. If Christopher had three cars, two trucks, and a ball. The unit bar would still represent the total number of toys (6), but it would be divided into 3 sections labeled "3 cars," "2 trucks," and "1 ball."

Note that in the examples above, all the facts are provided. No information is missing. Students first learn to use model drawing using complete problems like this where no information is missing. Students need multiple experiences organizing complete information onto the models before they tackle problems with missing information. Once they understand the relationship between the parts and the whole on the diagram, the problem can be changed to "Christopher has three cars and two trucks. How many toys does he have?" Students can then use a question mark or a variable like *X* to represent the missing information in their drawing, as illustrated by Example #1 in Figure 12.4.

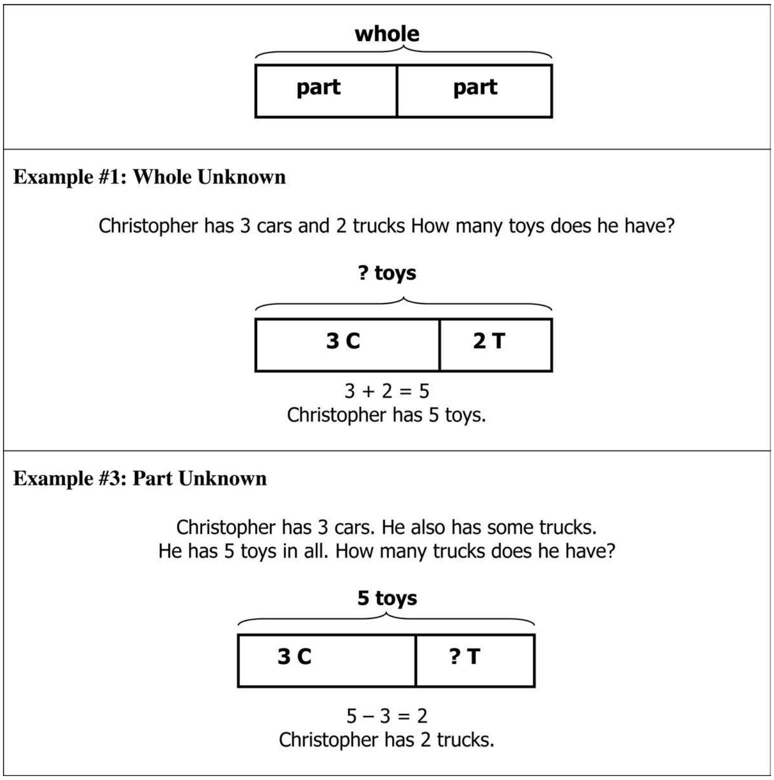
Using Schematic Diagrams to Develop the Equation

Schematic diagrams facilitate problem solution. The model visually demonstrates that the whole is equal to the sum of its parts. If the parts are given and the whole is missing, then you add the parts to obtain the whole. In the example above, Christopher has three cars and two trucks. A question mark, or variable such as X, represents the whole quantity. Through demonstration and discussion, the child learns that whenever the whole is missing, it can be found by adding the parts (3 + 2 = 5). This rule will apply to every schematic drawing where the whole is unknown.

If the whole is known but a part is missing, then it becomes a subtraction problem:

> Christopher has three cars. He also has some trucks. He has five toys in all. How many trucks does he have?

Figure 12.4 Model Drawing for Part/Whole Problems



See Example #2 in Figure 12.4. Again, through demonstrations and discussion the child learns that a missing part can be determined by subtracting the known part from the whole (5 – 3 = 2). Whenever the whole is known and one piece is missing, it is a subtraction problem. In Chapter 8, we discussed how unknown part problems can be solved by subtracting from the whole or counting up from a part. While both counting up from a part or subtracting from the whole yield the correct answer, problems that involve larger numbers are more easily solved using subtraction. Therefore, we recommend teaching students to use subtraction when solving word problems that involve solving for a missing part.

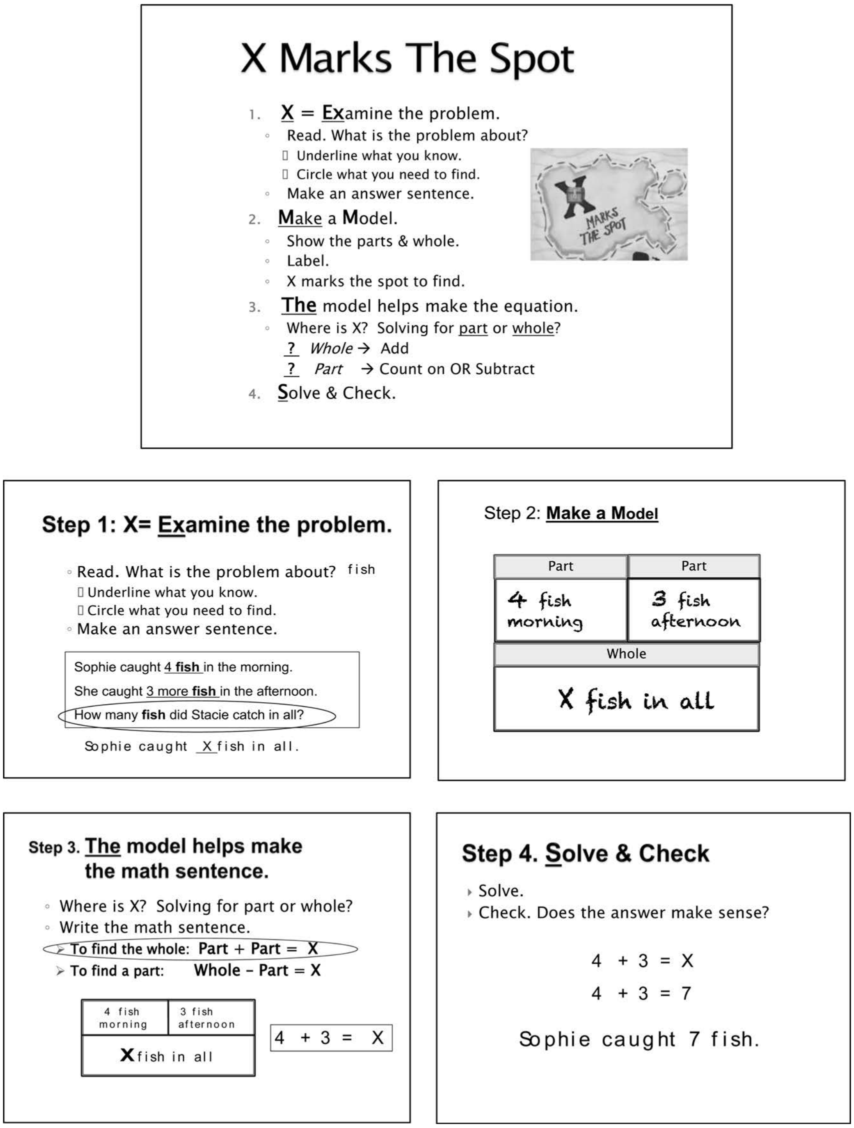
These two rules enable a student to solve any *part/whole* problem: (1) *to find the* *whole* *when the parts are given, add;* (2) *to find a* *part* *when the whole is given, subtract.* Explicitly teach students how to use these rules to solve part/whole problems.

Provide a Step-by-Step Strategy

Many students who struggle with mathematics have deficits in executive functioning, and teaching them to use a specific step-by-step strategy for each problem type can help them organize and monitor their work. Interventionists may find the *X Marks the Spot* strategy useful to supplement existing materials. Figure 12.5 shows the steps in the strategy as well as an example of how to use the steps to solve a *part/whole problem*.

The strategy uses a pirate treasure map to introduce the idea of problem-solving. Most children enjoy activities that include pirates, and most are familiar with the idea of placing an X on a treasure map to indicate the location of buried treasure. Teachers can expand on the pirate theme to engage and motivate their students. When solving a math problem, X

Figure 12.5 Using *X Marks the Spot* for Addition and Subtraction



will mark the spot where the solution to the problem can be found. The phrase "X Marks the Spot" is an acronym to help students remember the four steps.

- In Step 1, X cues students to "Examine" the problem. This is similar to Polya's first step: understand the problem. The students must read the problem and make sure they understand the topic and what they are trying to find. They are encouraged to underline what they know and circle what they need to find. Note that underlining and circling are also steps in the CUBES strategy. Identifying known and missing information is an important part of any problem-solving activity. This strategy differs from CUBES, however, because students do not use the information as a shortcut to immediately write an equation, which is what happens when using the ineffective keyword or CUBES strategies. Instead, they save the information and continue to work to more fully understand the problem. The final sentence in Step 1 says, "Make an answer sentence." Having students restate or rewrite the question in sentence form requires deeper reading that helps focus their attention on an appropriate solution. In the example provided, students identify that the problem is about fish, and recognize that they will have successfully solved the problem when they can fill in the blank in the sentence, "Sophie caught X fish in all." Younger children may say, "Sophie caught *blank* fish in all." Older children can insert an X into the sentence to represent the missing quantity.

- In Step 2, the letter *M* in "*X Marks the Spot*" reminds students to "Make a Model." In other words, they need to organize the information from the problem onto the schematic diagram. They must identify which information represents one or more parts, and which represents the whole. The numbers and labels that were underlined in Step 1 are now organized onto the diagram, and *X* is used to designate the missing information.
- Step 3 says, "The model helps make the math sentence." For many students, the most challenging part of solving word problems is trying to determine the appropriate equation to solve the problem. Materials designed for core instruction often omit teaching a student how to perform this critical step. Frequently, textbooks present students with a page of word problems that all require the same operation, so students do not have to understand the problem to determine which operation fits the problem scenario. Textbook lessons also focus on computational strategies, while providing minimal guidance about how to determine the correct operation. As a result, teachers say they look for other resources to help fill the gap. The internet is filled with sites that advertise keywords as a way to help students determine which operation to use for any given word problem. We have already discussed why keywords are ineffective, and how they actually interfere with students' ability to successfully solve more advanced word problems. The solution to this dilemma is to teach students to identify underlying structures. Once students organize the information from the problem onto the schematic diagram, they can easily identify the problem structures, and the solution equation becomes apparent. The rules for solving *part/whole* problems are, "*To find the whole when the parts are given, add. To find a part when the whole is given, subtract.*" Students can look on the diagram, note where *X* (the missing pirate treasure) is located, and determine whether they are solving for a part or for the whole. If *X* represents the whole, then it is an addition problem ("part + part = whole"). If *X* represents a part, then it is a subtraction problem ("whole – part = part"). This rule holds true for every type of part/whole problem students will encounter, whether they are solving simple problems like the one used in Figure 12.5, or problems involving multi-digit numbers, fractions, decimals, or algebra problems.
- In Step 4, the letter S in "Spot" cues the student to "Solve and Check." Students complete the computation, insert the answer into their solution sentence, and check to see if it makes sense. Checking can be done in many ways. It could involve simply answering the question about whether the answer makes sense or not, as shown in example in Figure 12.5. It could involve checking the answer using manipulatives or a calculator, comparing the answer to a previous estimation, or using an inverse operation to check the calculation. All of these are valid ways to check answers. Typically, only one approach would be used during a lesson.

The example provided in Figure 12.5 shows how to apply the *X Marks the Spot* strategy to solve an addition problem involving single digit whole numbers. The same strategy can be used to solve multi-digit subtraction problems where a part is missing. It is equally effective with fractions and decimals, and provides an excellent foundation for

Figure 12.6 Adding and Subtracting Decimals

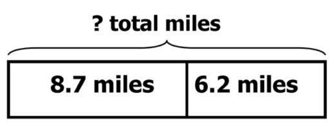
Example #1: Part/Whole Problem Involving Decimals

Angela bought jeans for $39.99 and a shirt for $24.99.
How much money did she spend in all?

$39.99 + 24.99 = $64.98
Angela spent $64.98 in all.

Example #2: Part/Whole Problem Involving Decimals

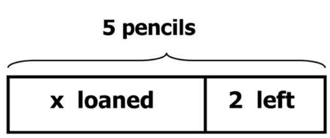
Kim drove 8.7 miles to her friend's house.
Then together they drove another 6.2 miles to the mall.
How far did Kim drive in all?



8.7 + 6.2 = 14.9
Kim drove 14.9 miles in all.

Example #3: Part/Whole Problem Involving Algebra

Juanita brought 5 pencils to school.
She loaned some of her pencils to friends, and now she only has 2 pencils left.
Write an equation for this problem.



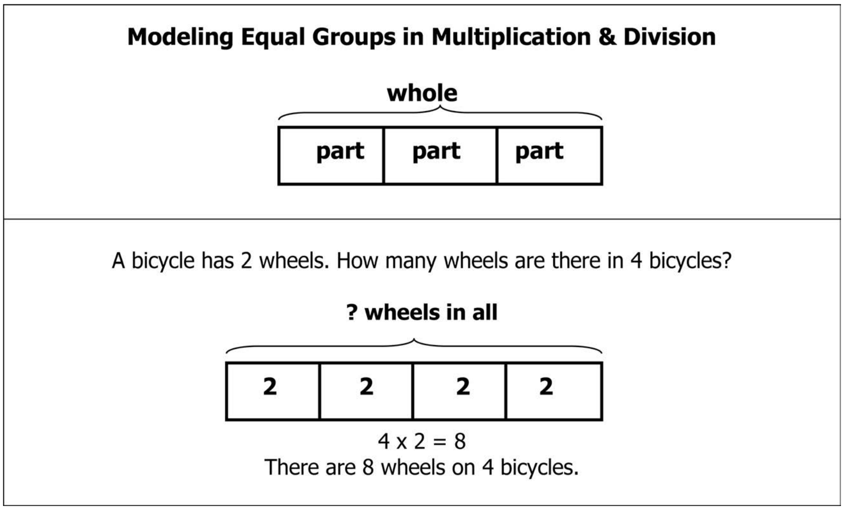
$x + 2 = 5$

understanding algebraic equations. See Figure 12.6 for examples of using tape diagrams with word problems involving decimals and algebra. Whether students use a bar model, as shown in Figure 12.5, or model the problem on a tape diagram, as shown in Figure 12.6, once they organize the information onto the schematic diagram, the underlying problem structure and appropriate equation are clear.

Multiplying and Dividing Part/Whole Problems

Multiplication is repeated addition, and model drawings clearly illustrate this relationship. In addition, the whole is equal to the sum of its parts; in multiplication, the whole is also equal to the sum of its parts. However, the parts in an addition problem represent addends (the numbers that will be added together) that have varying values. In a multiplication problem, the parts, or factors, represent equal sets. For example, an introductory multiplication problem might state:

Figure 12.7 Model Drawings for Multiplication and Division



A bicycle has two wheels. How many wheels are there in four bicycles?

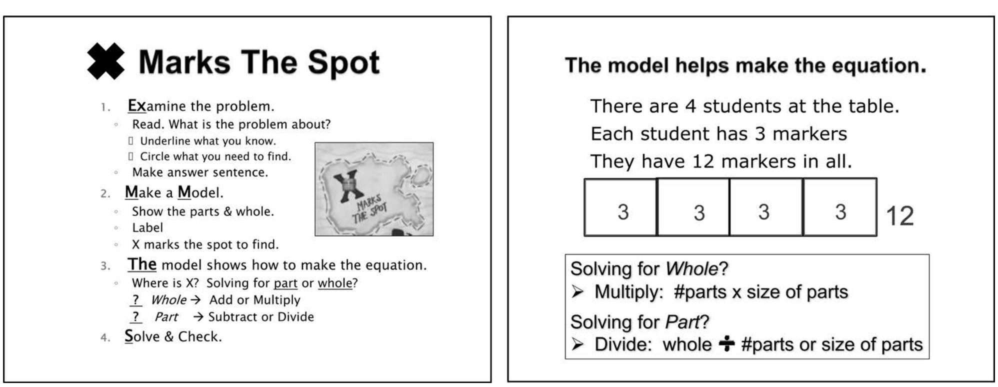
To illustrate this problem on a tape diagram, the student would draw a unit bar to represent the whole (i.e., the number of wheels on 4 bicycles). The unit bar would contain four boxes to represent the number of groups or sets (4 bicycles), with the number "2" written in each box to represent the two members of each set (the wheels on each bicycle). See Figure 12.7.

The drawing clearly shows that $2 + 2 + 2 + 2 = 4 \times 2 = 8$ wheels in all. The model drawing also demonstrates the relationship between multiplication and division, because the whole (8 wheels) is clearly divided into 4 sets of 2. The fact that division is repeated subtraction is also evident in the drawing.

The process for solving multiplication and division word problems is similar to that for solving addition and subtraction word problems, and the *X Marks the Spot* strategy applies equally well to multiplication and division problems. See Figure 12.8.

In a multiplication problem, the first factor indicates the number of groups and the second factor indicates the size of each group. In the problem shown in Figure 12.7, the four boxes represent the number of groups or parts, and the digit "3" written in each box indicates the size of each group/part. This diagram shows the equation $4 \times 3 = 12$ (i.e., four students each have three markers, so they have 12 markers in all). Because division is the inverse of multiplication, the diagram also shows that $12 \div 4 = 3$; 12 markers shared equally

Figure 12.8 Using *X Marks the Spot* for Multiplication and Division



among four students means that each student has three markers. As with addition and subtraction problems, it is helpful to introduce the strategy using problems where all the information is provided. Once students can effectively identify the parts and whole and organize the information on the schematic diagram, then they are ready to tackle problems that contain missing information. The location of X on the schematic diagram lets them know what operation to use. If X represents the whole, then it is a multiplication problem (part × part = whole). If X represents a part, then it is a division problem (whole ÷ part = part).

In Chapter 9, we discussed the fact that division can be interpreted two different ways: as *partitive* division or as *measurement* division. In partitive division, the divisor indicates the number of groups, and students must determine how many items are in each group. For example, if we buy a pizza that is cut into eight slices, we can create two different math problems. If we say, "There are four of us. How many slices can we each have?" then we have created a partitive division problem, because we know both the total number of slices and the number of groups/parts. The information that is missing is the size of each part. The tape diagram illustrated on the left in Figure 12.9 illustrates this partitive division problem.

In measurement division, the divisor represents the size of each group/part, so students must solve the problem to determine how many groups/parts they can make when they measure out groups of the given size. Using the pizza example above, if we say, "We each want two slices. How many people can we feed?" then we have created a measurement division problem. The figure on the right in Figure 12.9 illustrates how to represent a measurement division problem on a tape diagram. Since we do know how many groups we have, we cannot begin by drawing boxes on our diagram. Instead, we record the size of each group at either end of the tape diagram, and then use three dots to indicate that the pattern continues. We do not know how many times it will repeat, so that represents X, the question we need to answer. We can solve the problem with manipulatives or tally marks to measure out two slices per person, keeping track of how many groups we form before running out of slices of pizza. In this example, we form four groups before we use up all the pizza, so we know that the solution to the problem is $8 \div 2 = 4$. Students who have mastered the basic division facts can solve the problem abstractly, because they will already know the answer to this division problem.

Variations exist among the different authors who describe the process of model drawing.

Some books show problems modeled as described above, with the total value labeled with a bracket over the unit bar. Others use a similar model, but the bracket and total value are placed below the unit bar. Still others place the total value to the right of the unit bar.

Figure 12.9 Measurement and Partitive Division

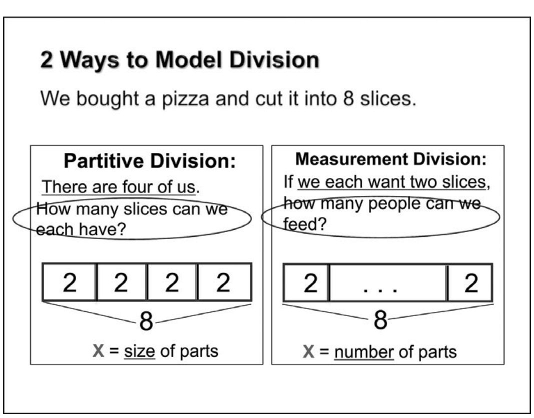
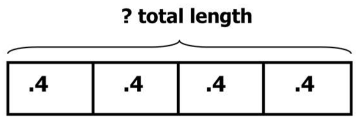


Figure 12.10 Modeling Multiplication and Division with Decimals

Example #1: Multiplication with Decimals

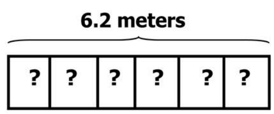
Joshua is building a birdhouse.
The plans say he needs 4 sections of wood that are each .4 meters long.
How long a piece of wood should he buy?



4 x .4 = 1.6
Joshua needs to buy a piece of wood that is 1.6 meters long.

Example #2: Division with Decimals

A piece of rope was cut into 4 equal pieces.
If the original rope was 11.2 meters long, how long is each piece?



11.2 ÷ 4 = 2.8
Each piece is 1.6 meters long.

Each variation effectively models word problems, and each one would help students understand the underlying pattern. Since students who struggle mathematically often become confused when they encounter slight variations in presentation, we suggest selecting one format and using it consistently in both core instruction and during tiered interventions.

The same schematic representations and *X Marks the Spot* strategy described for multiplying and dividing whole numbers is equally effective with decimal and algebra word problems. See Figure 12.10 for examples. For students with deficits in working memory or metacognition, connecting previous experiences using schematic diagrams and the problem-solving strategy to the problems they encounter in higher grades can significantly increase achievement outcomes.

Solving Additive Compare Problems

Compare problems have a different underlying structure than the *part/whole* problems discussed above. Instead of combining or separating parts of a single whole, as we did in the *part/whole* problems, additive *compare* problems involve two or more different items or sets of items. The problem may ask students to identify which is greater or lesser, taller or shorter, faster or slower, bigger or smaller, or any other comparison question. For example, in a *compare* problem the student might be given the ages of two children and asked who is older, or be given the weight of two objects and asked which is heavier, or the cost of three items and asked which costs the most. Unlike *part/whole* problems that address the parts of a single group or the quantity of the same items over time, *compare* problems always involve comparing two or more different items or sets of items. For example:

> Maria read four books. Her friend Jessica read five books. How many more books did Jessica read than Maria?

Figure 12.11 Examples of Words That Suggest a *Compare* Problem

Words That Suggest a *Compare* Problem	
more than / less than	bigger than / smaller than
taller than / shorter than	fatter than / thinner than
older than / younger than	longer than / shorter than
higher than / lower than	faster than / slower than

In this example the two items being compared are the quantity of books read by Maria and the quantity of books read by Jessica. The focus of the story is the comparison between different sets using a common unit (the number of books read). Another example of a *compare* problem is the following:

> James can lift a 100-pound weight. If his brother Marcus can lift 25 more pounds than James, how much weight can Marcus lift?

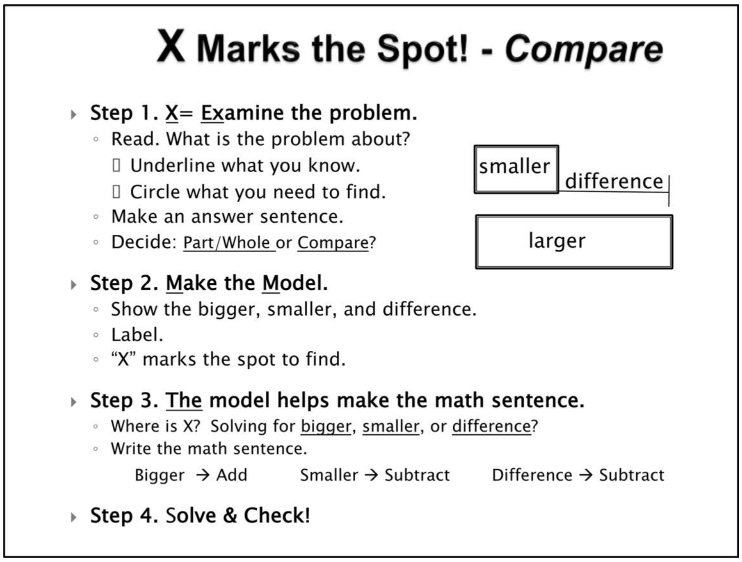
Two items are being compared: the amount of weight James can lift compared to the amount of weight Marcus can lift. *Compare* problems always involve a comparison between two or more different items or sets using a common unit of measure. Examples of comparison situations are listed in Figure 12.11. Note that although Figure 12.11 includes a list of words that signal a compare problem, this list is not the same a providing a list of keywords. Keywords are supposed to signal what operation to use. Because they do not work consistently and do not build understanding, experts urge teachers not to use keywords. The words in Figure 12.11 do not signal an operation. They simply alert the student that this is probably a *compare* type of problem.

Having students compare actual objects can help develop their understanding of comparison situations. For example, students can compare hand sizes with a classmate, or compare the lengths of their pencils. These concrete experiences help solidify their understanding of what it means to compare things. Students can build trains out of plastic cubes and then compare the lengths of their trains. They can play games using materials like "Mini Motor Math" or work with double number lines on their desks or on the floor. All of these concrete experiences build a foundation for later experiences solving word problems.

Compare problems can also be illustrated using hand gestures. Hold one hand in front of your waist, palm down, and say, "smaller." Hold your other hand above your chest as you say, "bigger." Finally, say "difference" as you move your hands up and down so that they alternate between touching each other and returning to their original extended position. Again, have students use this gesture whenever they encounter a *compare* problem. Moving hands up and down instead of side to side helps highlight the difference between *part/whole* and *compare* problems.

Because *compare* problems involve two or more different items or sets of items, they have different underlying structures and a unique schematic representation. In model drawing, each item or set of items in a *compare* problem is illustrated using a separate unit bar. A bigger bar is used to represent the bigger quantity and a smaller bar represents the smaller quantity. The difference between the length of the smaller bar and the length of the bigger bar represents the difference between the two quantities. A common error that students make when modeling *compare* problems on a schematic diagram is to draw the diagram the same way they show *part/whole* problems, with all the components combined in a single

Figure 12.12 Solving Additive *Compare* Problems



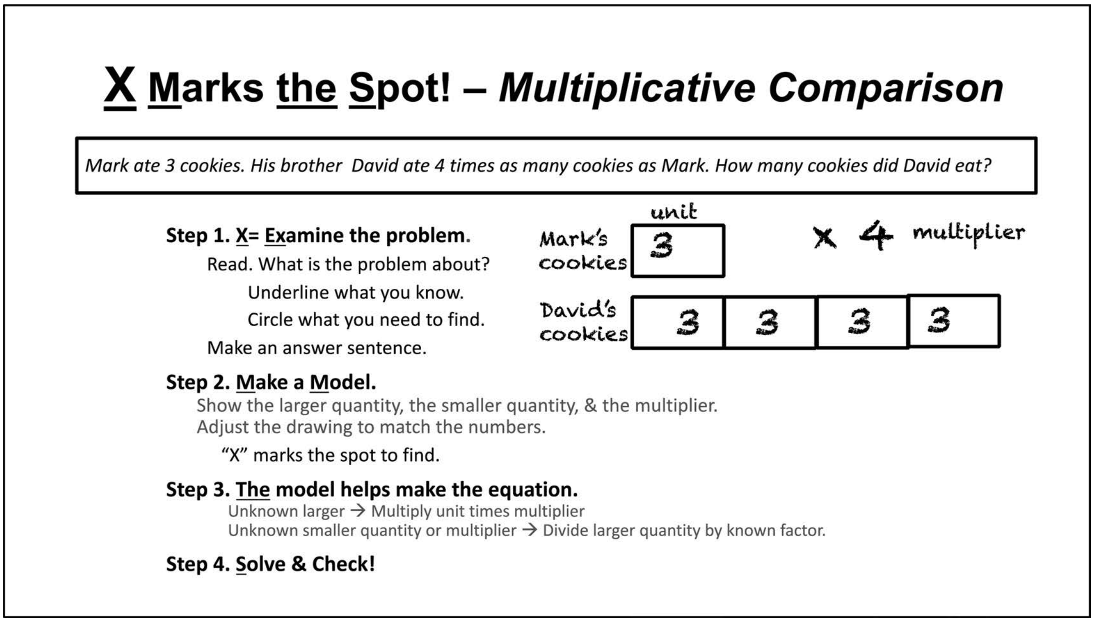
bar or tape. To accurately model the components in a compare problem, it is important to use two separate bars or tapes. The two quantities are not being combined to form one whole; they are discreet values that are being compared, so they need to be shown using separate bars or tapes, the same way they lay two objects next to each other to compare them. Figure 12.12 shows a model drawing for an additive *compare* problem.

The *X Marks the Spot* strategy works well to guide students through the process of solving *compare* problems. The steps are listed in Figure 12.12. The first step is the same as what students learned to use when solving *part/whole* problems, except that there is an additional question included at the end. Before students can model the problem on the schematic diagram, they must decide whether it is a *part/whole* problem or a *compare* problem. In a *part/whole* problem, they will model all the information in a single bar or tape, while a *compare* problem requires using separate bars to show the quantities being compared. Students should already be familiar with the words shown in Figure 12.8 that alert them to the possibility that this problem involves comparison. Step 2 is identical to what students have already done when solving *part/whole* problems. Before students are asked to solve for missing information in a *compare* word problem, they should practice organizing information from story problems that include all the information, just as they did when learning to solve *part/whole* problems. Once they are proficient at identifying the underlying structures in a *compare* problem, then they are ready to execute Step 2 for actual *compare* word problems. In Step 3, the rule for solving *compare* problems is similar to that used to solve *part/whole* problems, except that instead of using the "whole" or "total" quantity, you use the "bigger" quantity. If the bigger quantity is given, you subtract. If the bigger quantity is not given, you add. Finally, Step 4 is identical to the last step used when solving *part/whole* problems. See Figure 12.12. Students begin solving *compare* problems in first grade.

Solving *Multiplicative Comparison* Problems

In fourth grade, students are introduced to *multiplicative comparison* problems. These problems are similar to the *compare* problems in addition and subtraction because they compare two people or things using a common unit such as weight, age, etc. However, instead of the

Figure 12.13 Solving *Multiplicative Comparison* Problems



comparison words used in addition and subtraction such as "more than," or "less than," multiplicative comparison problems use words such as "three times as many," "twice as large," "four times more," or "five times as much" to describe the relationship between the two sets. Therefore, they have a unique underlying structure and are modeled with a different schematic diagram. The components of a multiplicative comparison problem include a larger quantity, a smaller quantity, and a multiplier. The smaller quantity also represents the "unit." (In higher grades, students also tackle problems where the multiplier is a fraction, such as "1/2 as large." Then the larger quantity would be the unit.)

Students can use the *X Marks the Spot* strategy to solve multiplicative comparison problems. See Figure 12.13.

We will use the following problem as an example of how to use the strategy.

> Mark ate three cookies. His brother David ate four times as many cookies as Mark. How many cookies did David eat?

To illustrate this problem, students must first identify what is being compared. In this example, the number of cookies Mark ate is being compared to the number of cookies David ate. That is what the problem is about. Students would complete Step 1 by underlining the information in the story, and then circling the question and creating an answer sentence. The answer sentence for this problem would be, "David ate *X* cookies." In Step 2, students would draw a bar to represent the unit (in this example, the number of cookies eaten by Mark) and label it (in this example, the label would be "3" or "3 cookies"). Since David ate four times as many cookies as Mark, the multiplier is ×4. That means that the student would represent David's cookies by creating a bar containing four units. Since each unit is worth three cookies, the number "3" can be written in each box of the bar showing David's cookies. See Figure 12.13. Once students complete Step 2, they can use the model to develop an equation by following the guidelines in Step 3. If the larger quantity is unknown, then multiply the unit times the multiplier. If the smaller quantity or the multiplier is unknown, then divide the larger quantity by the known factor. In Step 4, students solve the problem and check their answer. The easiest way to check the answer is simply to refer back to the schematic diagram created in Step 2. Students can check their solution by plugging the answer into the diagram.

Following a step-by-step strategy helps students monitor their work and so builds independence. Instead of relying on a teacher to remind them what to do next, the student can check off each step as it is completed. This should not be a rote procedure, however. To build deep understanding, encourage students to explain what they are doing and why they are doing it that way.

Solving Two-Step Problems

In this chapter, we have described how to use schematic drawings and an explicit strategy to solve a variety of word problems. For each problem type, our discussion focused on one-step problems that involved a single calculation. After students have mastered one-step problems, they progress to solving two-step problems that require two separate calculations. By the end of second grade, the Common Core State Standards specify that students to be able to solve one- and two-step problems involving addition and subtraction within 100, and by the end of third grade the they should be able to solve two-step word problems using all four operations (National Governors Association Center for Best Practices, 2010). Here is an example of a two-step problem:

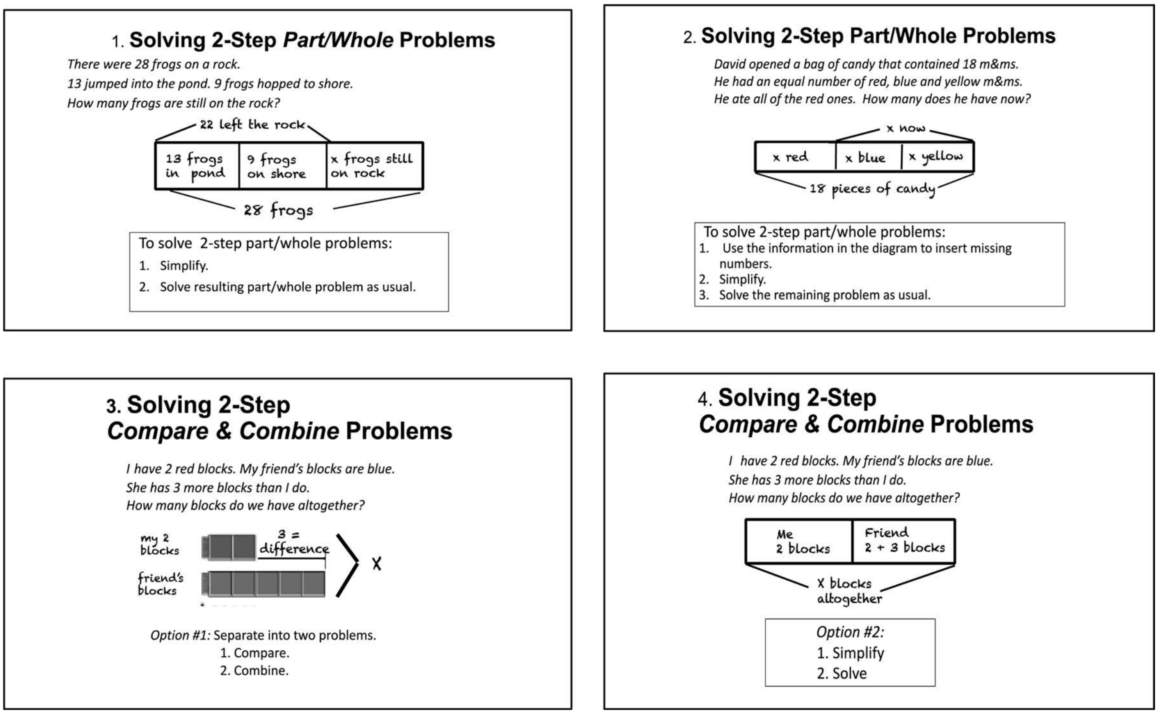
> Alex had $10. He bought a movie ticket for $7 and a soda for $2.50. How much money does he have left?

To solve this problem, the student must first determine the total amount of money Alex spent ($4 + $2.50 = $9.50), and then subtract that amount from the money Alex had originally in order to determine how much he has left ($10 – $9.50 = $.50). The solution requires two separate calculations, first to determine how much money Alex spent, and then to determine the amount of money left after he made his purchases. In second grade, both steps in the two-step problem usually involve part/whole problems, such as the example provided above. In higher grades, students are often asked to solve a problem that involves first comparing quantities, and then combining the results to answer the question posed in the problem. We will discuss each of these problem formats separately.

Two-Step Part/Whole Problems

These problems are easier for students, and are generally introduced first. The first two steps for solving these problems are similar to the steps provided in the *X Marks the Spot* strategy for all part/whole problems. Following Step 1 of the strategy, students examine the problem and identify the information they know and what they need to find. For Step 2, they "Make a Model," (i.e., organize the information on the graphic organizer). See the first example in Figure 12.14. Once they have created a schematic diagram to organize the information in the problem, they need to develop equations to solve the problem. This is where a two-step problem differs from the part/whole problems they have encountered before. To answer the question posed in the problem, students must first simplify the problem (i.e., combine similar information). In the example in Figure 12.14, this means combining the 13 frogs that jumped into the pond with the 9 frogs that hopped onto the shore. This is a standard *part/whole* problem: part + part = whole. They add the parts to determine the total number of frogs that left the rock: 13 + 9 = 22. Once they have solved the first *part/whole*

Figure 12.14 Solving Two-Step Problems



problem, they are ready to tackle the second part of the two-step problem: If there were 28 frogs on the rock and 22 left the rock, how many are still on the rock? Students working on 2-step problems should have already mastered the rule that "whole – part = part," and so should be able to create an appropriate equation to solve this problem: 26 – 22 = 4.

The second example in Figure 12.14, illustrates a more challenging problem, because some numbers were not provided. Students should approach this problem as they do all word problems by first examining the problem and understanding all the information, and then organizing the information on a schematic diagram. In other words, they follow the first two steps of the *X Marks the Spot* strategy. They can then use the schematic diagram to develop their first equation. In this example, students can see that they have created a standard *part/whole* division problem: whole ÷ part = part. The diagram shows that 18 pieces of candy were split into equal groups, so there would be six pieces of candy in each group: 18 ÷ 3 = 6. Students can write 6 in each box to indicate how many red, blue and yellow m&ms David had. Once they have completed that portion of the problem, they are ready to tackle the remaining *part/whole* problem and answer the question stated in the problem: "How many [pieces of candy] does he have now?" This is another familiar *part/whole* structure: whole – part = part. The whole bag of candy, minus the red pieces David ate, tells how many m&ms are left: 18 – 6 = 12. David has 12 pieces of candy left.

Two-Step Compare and Combine Problems

Starting in third grade, students begin to encounter *compare and combine* problems. The process for solving these problems is similar to the process described above, except that the first part of the problem is a *compare* problem, and the second is a *part/whole* problem. We describe two options for solving this type of problem. In one approach, students start by creating a typical compare schematic diagram and solve that portion of the problem, then solve the resulting *part/whole* problem. See the third example in Figure 12.14. The example shows a *compare* model when the larger quantity is unknown. Before being introduced to two-step *compare and combine* problems, students should already know the rule for solving *compare* problems: "smaller + difference = larger." Therefore, for this problem, they should be able to write the equation 2 + 3 = 5, and recognize that the friend has five blocks. Next, students solve the *part/whole* problem by combining the larger and smaller quantities: my 2 blocks + friend's 5 blocks = 8 blocks in all. shows A second option for modeling and solving the same problem is illustrated in the final example in Figure 12.4. More sophisticated students may recognize that they can simplify the modeling process if they immediately treat this as a part/whole problem. These students recognize that the friend has 2 + 3 = 5 blocks, and want to record it that way on their model, as shown in the final example. "My" 2 red blocks represent one part of the whole, and the friend's 2 + 3 blocks represent the other part. These two parts combine to form the whole. Students who are ready to model the problem this way should be encouraged to do so.

When two-step problems are first introduced, students will need many opportunities to practice discriminating between the *part/whole* or *compare* problems they have already learned to solve, and the two-step problems they now must master. Students should be able to consistently recognize two-step problems before they are asked to solve them. Because two-step problems have more steps, they require additional working memory. Students who have deficits in working memory or executive functioning will find these problems particularly challenging, and may need extensive practice opportunities before they can solve them proficiently. Encouraging students to continue to use the hand gestures, schematic diagrams, and step-by-step strategy they employed when solving one-step word problems. Comparing and contrasting the various problem types, and discussing how the strategies for solving them are alike and how they differ, will help students develop a deeper understanding of problem-solving.

Intensify Instruction

Interventionists who have access to a program that is validated for use with students receiving Tier 2 supports should implement the program with fidelity. If a validated program is not available, then the interventionist will need to intensify the instruction provided in the textbook or available resources by adding or increasing use of the strategies described in this chapter. It is not enough to simply follow the program provided during core instruction, because that has already been shown to be ineffective for the student. Instead, add additional evidence-based supports to intensify instruction. If a student continues to struggle after receiving Tier 2 support, then that student needs even more intensive support during Tier 3 interventions. If a validated program was used during Tier 2 instruction, intensify it further by increasing use of the supports described below. If a validated program was not used for Tier 2 instruction, increase the intensity of instruction the same way, by increasing use of the supports described below.

- ♦ Follow the C-P-A continuum. Students' initial experiences with problem-solving should be at the concrete level, using their bodies and concrete objects to act out story problems. Schematic diagrams can be introduced after students have developed conceptual understanding at the concrete level. Students who struggle with mathematics learn more easily when the connection between the concrete and more abstract representations is explicitly demonstrated.

- ♦ Teach students to recognize the underlying schematic structures in word problems, as described above. Most basal programs introduce a mixture of problem types without providing clear guidance to help students recognize the semantic structure of the problem. Research studies show the benefits of teaching one type of story problem at a time. When faced with word problems, students who have experienced difficulty in mathematics tend to jump to computation before they fully understand the problem. If the first problems they encounter contain complete stories with no information missing, they can focus attention on the semantic structures in the problems, rather than on racing to find a solution. Problems with missing information can be introduced after students learn to recognize the underlying semantic patterns. When instructors first introduce a problem type, all examples should illustrate that structure. Diverse story elements can be gradually introduced so that students learn to discriminate among types of problems.
- ♦ Explicitly teach students how to use schematic diagrams. These graphic organizers provide non-linguistic representation of the facts in a problem and the relationships among those facts. Students need to be taught how to organize the information onto these diagrams. Begin by demonstrating the relationship between students' concrete experiences and the graphic organizers. Students who struggle with problem-solving may grab numbers from a problem in the exact order they appear in the story. Providing clear, consistent modeling followed by guided practice can ensure that students create models that reflect an accurate understanding of the story.
- ♦ Provide an explicit strategy. Problem-solving is a complex task that requires multiple steps to be accurately executed in a particular sequence. Students with deficits in metacognitive reasoning tend to have difficulty remembering and executing steps in sequence, and so benefit when taught an explicit strategy that provides step-by-step guidance. Because these students often fail to notice whether their calculations are accurate or their answer makes sense, they need to be taught to evaluate their work and this should be included as one of the steps in the strategy. A checklist of strategy steps provides a useful visual prompt that supports independence. The checklist can be systematically faded once students can successfully execute the strategy steps.
- ♦ Carefully sequence problems from simple to complex. The current trend in mathematics instruction is to present students with complex problems that have multiple entry points. Some students find this complexity overwhelming, and they are unable to focus on the salient features of the problem, discern patterns among problems, or generalize solution strategies from one example to the next. Students who find problem-solving an uncomfortable challenge benefit when problems are carefully sequenced. They need to experience success with routine problems first, with non-routine formats introduced after they have demonstrated proficiency with the simpler problems. Examples of the kinds of changes that should be systematically introduced include changing the order in which information is presented, adding unfamiliar vocabulary or complex sentence structure, introducing irrelevant information, presenting information with charts, graphs, diagrams or other visual representations, and providing problems that require multi-step solutions. These changes should be systematically and explicitly introduced, discussed and practiced so that students realize superficial changes do not change the underlying story structure.
- ♦ Follow the explicit instruction model. For each new concept or skill step, provide teacher-mediated instruction followed by guided practice followed by independent practice. Students who struggle with mathematics learn best when each step is explicitly taught. Once students demonstrate proficiency with one skill, they are ready to tackle the next

skill in the sequence. While some students are able to progress rapidly from simple to more complex problems, others may need multiple experiences with each skill step before they are ready to move on to the next step in the sequence.

- ♦ Have students explain what they are doing, and why they are doing it this way. Asking students to explain their reasoning helps them solidify understanding, and also provides valuable formative assessment information that can be used to refine instruction. Core curriculum materials increasingly stress the importance of communication in mathematics. Too often students receiving interventions have learned to use tricks and follow steps by rote, without developing conceptual understanding. Asking students to explain their own reasoning, and to understand and critique the reasoning of others, is important to develop mathematical proficiency.
- ♦ Systematically fade support. Students who receive math interventions need scaffolded support to gain proficiency, but the ultimate goal of tiered support is for them to function successfully in the regular classroom. They need to learn to handle complex instruction with information introduced in bigger chunks. When instructors gradually fade support while continuing to monitor performance, students are able to transition more easily back into the regular instruction.

Summary

In this chapter, we have provided suggestions for improving problem-solving skills among students who struggle with mathematics. Teaching students to identify underlying structures in word problems, and then use those structures to solve the problem, is an evidence-based practice that has been shown to significantly improve problem-solving ability (Jitendra et al., 2015; Pfannenstiel et al., 2015; Powell & Fuchs, 2018). We have described the structures of various types of word problems and explained how students can organize the information from the story into a schematic diagram. We have also introduced a step-by-step strategy to help students use the information from the diagram to identify the appropriate equation and solve the problem. Because the evidence supporting this approach to problem-solving is so strong, the What Works Clearinghouse recommends it for all students receiving tiered interventions:

Teach students about the structures of various problem types, how to categorize problems based on structures, and how to determine appropriate solutions for each problem type. (Gersten et al., 2009, p. 27)

Implementing the suggestions described in this chapter can enable all students to become proficient problem-solvers.

13

Conclusion: Using RtI to Improve Achievement in Mathematics

Almost two-thirds of American students are not achieving expectations for mathematics. Results of the 2019 National Assessment of Educational Progress (NAEP), often called the "nation's report card," show that only 21 percent of 12^{th} grade students are proficient in mathematics, while 40 percent of students scored below the basic level (NCES, 2019.). Clearly, change is needed if our students are to be mathematically competent and our nation is to remain globally competitive.

Response to Intervention is a comprehensive school improvement model designed to help all learners achieve academic proficiency. The core elements of RtI include: (1) providing high-quality instruction for all students to prevent mathematics difficulties, (2) using data to guide instructional decision making and evaluate instructional effectiveness, and (3) providing support for students who are at risk for academic failure by providing multiple levels of increasingly intense, targeted interventions. In this final chapter, we will provide a brief review of how the RtI/MTSS framework can improve mathematics achievement for all learners and provide suggestions for locating the evidence-based instructional materials and strategies necessary to successfully implement RtI.

Selecting Materials for Core Instruction (Tier 1)

The core curriculum is the instructional program provided in the general education classroom. In the most recent tests of national achievement, only 34 percent of U.S. eighth-grade students and 41 percent of fourth-grade students scored at or above the proficient level in mathematics (NCES, 2019). U.S. students' mathematical performance lags when compared to other industrialized nations (OECD, 2019). One reason our current approach is not achieving the desired outcomes may involve inadequacies in the textbooks and other instructional materials we are using. A 2019 research report prepared by the Center for Education Policy Research at Harvard University found that 93 percent of educators use textbook lessons for most mathematics instruction (Blazer, Heller, Kane, Poliko, Staiger, Carrell, & Kurlaender, 2019), and another summary of mathematics education in the United States reported that the majority of school systems rely on a single textbook (Dossey, Soucy McCrone, & Halvorsen, 2016). The data suggest that textbooks have a major role in how instruction is provided, but available instructional tools and textbooks often do a poor job of adhering to important instructional principles for teaching and learning mathematics (Alkhateeb, 2019; NMAP, 2008).

The RtI model calls for using evidence-based instruction with all students. To help educators locate effective practices and materials, the U.S. Department of Education and several other organizations review pertinent research studies and publish their findings. Figure 13.1 provides information about accessing resources for core instruction (Tier 1) as well as interventions (Tiers 2 & 3).

While most publishers advertise that their products are supported by research and will lead to significant academic growth, existing research does not support most of these claims. A review of the programs reviewed in the online resources shows a limited number of available programs designated as evidence-based. While many studies of commercial programs are conducted, very few use the rigorous methodology necessary to convincingly demonstrate that the program leads to improved achievement outcomes for students. This does not mean that all these programs are ineffective. It does mean that districts must often purchase instructional programs without the benefit of adequate data to inform their decisions.

Although high-quality research evaluating complete math programs is currently sparse, a large body of research has identified instructional design features that can produce significant improvements in achievement. The National Council of Teachers of Mathematics has posted a list of questions to consider when adopting core curriculum materials. It is available at https:www.nctm.org/News-and-Calendar/Messages-from-the-President/Acrhive/Diane-Briars/Curriculum-Materials-Matter_-Evaluating-Evaluation-Process/. In Chapter 3, we provided a list of recommended evidence-based instructional design features that should be present in any program selected for core instruction. We have used these recommendations to create additional questions to use when selecting instructional programs for use as a core curriculum, and these questions are available in the online materials. The features included in this checklist do not include all the factors a district might want to consider when selecting core materials, such as how the materials portray diversity, the availability of support materials for teachers, program costs, and so on. This list focuses on instructional factors that have been shown to improve achievement outcomes through multiple, rigorous scientific studies in regular education settings. Because the instruction provided in the core curriculum should be responsive to the needs of *all* students, materials selected for core instruction should also include instructional procedures that are critical to support students who are at risk. Districts that have a high percentage of students failing to meet benchmark expectations should place special emphasis on these additional characteristics when selecting materials for the core curriculum.

A cornerstone of RtI/MTSS is the use of data to evaluate instructional effectiveness and to guide instructional decisions. When schools follow an RtI/MTSS model, all students are screened two or three times per year. Screening provides districts with a wealth of data to evaluate the effectiveness of their core curriculum. A program is generally considered effective if at least 80 percent of students consistently perform at benchmark on screening measures. Note that even when learners experience high-quality instruction, up to 20 percent of students may need additional support in order to be successful. If a district finds that 80 percent or more of the students consistently perform well on screening measures, then it may be reasonable to assume that the curriculum is effective. If less than

Figure 13.1 Internet Resources for Locating Evidence-Based Materials.

Internet Resources for Locating Evidence-Based Materials

- **Best Evidence Encyclopedia**
  Center for Data-Driven Reform in Education (Johns Hopkins University)
  http://www.bestevidence.org/
  The Best Evidence Encyclopedia (BEE) reviews educational programs and provides an overview of the program, evidence of effectiveness ratings, and contact information for obtaining the materials. The site includes programs for use in the core curriculum and programs that are useful for interventions

- **Center on Instruction**
  RMC Research Corporation
  http://www.centeroninstruction.org/
  The Center on Instruction (COI) is funded by the U.S. Department of Education. It develops and identifies free resources that to help educators provide high quality education. The site does not evaluate specific math programs, but does provide articles, modules, practice guides, and archived WebEx resources on progress monitoring and math interventions.

- **National Center on Intensive Intervention**
  American Institutes for Research
  https://intensiveintervention.org
  The National Center on Intensive Intervention is a leader in supporting schools, districts, and states integrating MTSS. The site includes an Academic Intervention Tools Chart that provides a detailed description of programs to use in math, reading and social-emotional interventions, and lists the study or studies used to evaluate the program, the quality of the study, the effect size, and costs and training required to implement the program

- **Promising Practices Network**
  RAND Corporation
  http://www.promisingpractices.net/programs.asp
  The Promising Practices Network (PPN) provides descriptions of programs that have been proven to improve outcomes for children. All programs featured on the site have evidence of positive effects in rigorous scientific studies. This site provides information on educational programs for students in the core curriculum and those with special needs. To locate programs related to math outcomes, follow the link for "Cognitive Development/School Performance."

- **What Works Clearinghouse**
  U.S. Department of Education Institute of Education Sciences
  http://ies.ed.gov/ncee/wwc/
  The What Works Clearinghouse (WWC) website states that its mission is "to be a central and trusted source of scientific evidence for what works in education." They review and synthesize the research, with the goal of providing educators with "the information they need to make evidence-based decisions." The site contains a variety of resources in addition to program reviews. To locate evidence-based materials, go to "Publications and Reviews," click on "Find What Works!" and follow the directions provided.

80 percent of the students are meeting instructional benchmarks, then the district should evaluate the core program to ensure that the program being used is evidence-based, that it is being implemented with fidelity (i.e., delivered for a sufficient time and in the manner intended), and that adequate resources are available to support effective instruction. The core curriculum should match the characteristics of the learners being served. The same program may work well in one district and be less effective in another due to differences in learner characteristics. If a high percentage of students are not meeting benchmark expectations, a district might consider selecting core materials that place greater emphasis on the evidence-based instructional practices recommended for use with students at risk for academic failure.

Analyzing detailed assessment results, rather than simply considering overall scores, can also yield valuable insights. It may be that students score well on most subtests, but a high percentage of students in multiple classrooms perform poorly in one particular area. Based on analysis of the data, a district might decide to devote more time to that topic or provide teachers with additional training to enable them to teach that particular content more effectively. The district might also choose to supplement the curriculum in that area. Comparing assessment results across years can also yield valuable insights, because the results obtained using a particular program can vary over time depending on the specific group of students being served. Since students' background knowledge and mastery of skills can vary from year to year, the core math curriculum may adequately meet students' academic needs in some years, while in other years, supplementary instruction or materials should be added in order to help students achieve benchmark expectations. By using data to evaluate instructional effectiveness, districts can fine-tune their curriculum to meet student needs.

Selecting Materials for Interventions (Tiers 2 & 3)

Even when learners experience high-quality instruction, it is possible that up to 20 percent of students will need additional support in order to meet benchmark expectations. Careful screening enables educators to detect problems early and provide timely interventions. RtI/MTSS uses a tiered service delivery model to efficiently distribute instructional resources in order to provide early intervention to the greatest possible number of students. Individuals who are not making adequate progress in the core curriculum receive supplemental support through multiple tiers that provide increasingly intensive interventions. These interventions should be targeted to match student needs and should employ evidence-based intervention strategies. Students can move in either direction between tiers or, if appropriate, go directly to the most intensive intervention level. Student progress is monitored carefully throughout these interventions, and the data are used to adjust instruction to increase learning outcomes.

Tier 1 represents the general classroom where the core instructional curriculum is delivered. In Tier 1, instruction should be differentiated to provide additional support for learners identified through the universal screening as not having mastered the core curriculum objectives. Therefore, the materials selected for use in core instruction should provide teachers with guidance and resources to help teachers differentiate instruction to support students when they struggle with core content.

If differentiating instruction is not sufficient, a student should receive Tier 2 support. In Tier 2, interventions are provided to small groups of three to five students who share common instructional needs. Students receive about 120 minutes per week of mathematics intervention in addition to the math instruction provided in the regular classroom. Interventions should focus on the essential content needed by the students in the group, as identified through universal screening, ongoing progress monitoring, and diagnostic assessments, and should employ evidence-based instructional strategies. Many publishers offer math programs that they say are designed for use in Tier 2 interventions. Some of these are "validated programs," which means "there is positive evidence, collected during at least one well-conducted randomized control trial, that the program improves the mathematics outcomes of students with MD (mathematical disabilities) in a Tier 2 intervention" (Powell & Fuchs, 2015, p. 183). If a program is validated, then it is appropriate to use during Tier 2 interventions. The National Center on Intensive Intervention provides an Academic Intervention Tools Chart (see Figure 13.1) which summarizes efficacy studies of mathematics intervention programs. This chart can assist educators in finding validated intervention materials. However, many teachers who provide Tier 2 support do not have access to a validated program that meets the needs of their students (Powell & Fuchs, 2015). If an interventionist does not have access to a validated program, then the available resources must be intensified in order to incorporate appropriate Tier 2 interventions (National Center on Intensive Intervention, 2020). We discuss ways to intensify instruction later in this chapter.

Students who continue to struggle after receiving Tier 2 support should obtain more intense interventions through Tier 3 support. (Note that while most states follow a three-tier model, states vary in the number of tiers they use and the point at which they begin the special education referral process.) In Tier 3, students receive "individualized support." In this context, the term "individualized" does not mean that instruction must be provided in a one-on-one setting. The term "individualized instruction" or "specially designed instruction" means that instructional objectives and methods are individualized to meet the unique needs of the learner. In other words, the instruction provided at Tier 2 should be further intensified for students who receive Tier 3 support. If a validated program was provided in Tier 2, then Tier 3 interventions may be developed by building on the existing program. If a validated program was not used in Tier 2, then Tier 3 interventions must be developed by further intensifying whatever materials were used during Tier 2 (McInerney, Zumeta, Gandhi, & Gersten, 2014).

Because students differ in the supplemental support they require, intervention materials are not "one size fits all." The IES Practice Guide recommends that interventions for students in grades K5 focus intensely on in-depth treatment of whole numbers, while interventions for students in grades 4-8 focus on rational numbers and advanced whole number topics (Gersten et al., 2009). However, as the authors point out, older students who have not mastered whole numbers may need to spend additional time on prerequisite skills involving whole numbers before they are prepared to tackle rational numbers. Just because students are the same age does not mean they need the same intervention support. Some publishers advertise RtI materials for specific grade levels, such as "RtI for fifth grade" or "RtI for your third-grade intervention students." These generic materials ignore the importance of data-based decision making. Students who receive tiered support should be grouped so that they share similar instructional needs, and then materials can be selected to address those particular needs. This requires that districts first locate instructional materials founded on evidence-based practices and then use those materials selectively by matching materials to students' identified needs. Districts may need to purchase a variety of programs and materials to support the range of needs present in the students they serve.

Intensifying Instruction

When the available materials are not specifically designed for students with math disabilities, then interventionists will need to make adaptations to intensify them. “Intensifying instruction” means adapting the existing program to more effectively address a student’s targeted needs.

Interventionists who have access to a program that is validated for use in Tier 2 should implement the program with fidelity. If a validated program is not available, or if a student continues to struggle after using a validated program, interventionists can use the strategies below to intensify interventions.

1. Focus on foundational skills. Because understanding whole numbers and rational numbers forms the foundation for all other mathematics, it is recommended that intervention time focus on these foundational concepts. Intervention time is limited, so interventionists should spend that time on the content that will have the greatest impact on achievement (Gersten et al., 2009).
2. Build fluent retrieval of basic facts. Research has shown that automaticity with basic facts predicts performance on general mathematics tests (Stickney, Sharp, & Kenyon, 2012), and that students who struggle in mathematics typically lack automaticity with basic facts (Baker & Cuevas, 2018; Gersten et al., 2009). Therefore, the IES Practice Guide recommends, “Interventions at all grade levels should devote about ten minutes in each session to building fluent retrieval of basic arithmetic facts” (Gersten et al., 2009). The guide further recommends focusing on two unfamiliar facts per session (Gersten et al., 2009).
3. Use systematic instruction. Select objectives carefully and sequence them from easiest to hardest, making sure that prerequisite skills are mastered before introducing more complex content. If students struggle, objectives can be further broken down into component parts or steps. If a student struggles to complete all the steps in a single lesson, then the lesson could be broken down to focus on only one or two steps each day. Although it will take longer to introduce the complete procedure, this approach often saves time in the long run because it reduces the need for re-teaching. To avoid overwhelming students’ cognitive capacity, pace instruction so that students solidify their understanding of one concept or skill before introducing another (Powell & Fuchs, 2015).
4. Use explicit instruction. Follow the guidelines described in Chapter 5. If the available materials do not use this high-leverage practice, then modify the lesson to include all the elements of explicit instruction (Gersten et al., 2009, McLeskey et al., 2017).
5. Model additional examples. Students who struggle with mathematics benefit from seeing multiple models of effective problem-solving. Core materials may not provide sufficient examples before asking students to try to solve a problem themselves. Providing multiple examples supports student learning (Gersten et al., 2009; Powell & Fuchs, 2015).
6. Provide additional think-alouds and have students explain their reasoning. Talk aloud as you model how to solve a problem, so students can understand your thought process. Providing extra think-alouds supports student understanding and also provides a model that students can follow when asked to explain their own reasoning (Gersten et al., 2009; Powell & Fuchs, 2015).
7. Give students a written list of steps to follow, and teach them to refer to the list and check off steps as they complete them. Many students who struggle with mathematics have deficits in executive functioning. Teaching self-regulation is an evidence-based

practice that has been shown to increase achievement (The IRIS Center, 2020; McLeskey et al., 2017).

8. Emphasize academic vocabulary. Explicitly teach mathematical terminology, then emphasize and repeat the language in subsequent lessons (Powell & Fuchs, 2015). Adding gestures supports working memory and helps students make connections, which can improve academic achievement (Hord et al., 2016; Walsh & Hord, 2019).
9. Increase the use of concrete and pictorial representation. Students who struggle with mathematics often have difficulty understanding abstract symbols (van Garderen et al., 2018). Studies show some may need as many as 60 problems where they use concrete objects to model problems, and 60 more examples using two-dimensional models such as drawings and diagrams, before they are ready to rely solely on abstract words and symbols (Butler et al., 2003; Miller & Hudson, 2007). Textbooks generally move quickly to abstract representation. Increasing the use of concrete and pictorial representation can increase achievement (Gersten et al., 2009; Powell & Fuchs, 2015).
10. Link differing ways of representing a problem. Every problem can be modeled multiple ways, and exposing students to multiple representations helps strengthen understanding. Textbooks provide a variety of representations, but they sometimes use them in isolation, and students who struggle with mathematics may not recognize how the various representations are related. Support learners by connecting and comparing the various representations included in textbook materials (Gersten et al., 2009).
11. Provide positive and constructive feedback. Feedback can be verbal, nonverbal, or written. Effective feedback is timely, genuine, and provides guidance that helps the learner improve performance (McLeskey et al, 2017).
12. Make judicious use of computer-based instruction (CBI). A multitude of computer programs advertise that they will improve students' mathematical performance. Hawkins et al. (2017) identified characteristics found in CBI programs that effectively develop mathematical proficiency. They include: (1) customization features that allow the instructor to individualize practice, (2) ample opportunities to respond, (3) immediate feedback and error correction, and (4) progressive monitoring features. Intervention time is valuable, and instructors must choose wisely to provide the type of focused practice that supports students who require mathematical interventions.
13. Include motivational strategies. Studies show that addressing students' motivation, especially with the use of structured rewards, can have a greater impact on mathematical achievement than the choice of textbooks or the provision of computer-assisted technology (Best Evidence Encyclopedia, 2020). These findings have led experts to recommend that mathematical interventions should include a motivational component (Gersten et al., 2009; NMAP, 2008).

In the online resources, we provide a list of questions that can be used to assess how well any instructional material incorporates elements that have been found effective during mathematical interventions. These questions may help an interventionist to identify missing components, and so guide the selection of adaptations to intensify existing materials.

We discussed a variety of methods for intensifying instruction throughout this book. Additional resources to help interventionists meet their students' needs include the IRIS module, "Intensive Intervention Part 1: Using Data-Based Individualization to Intensify Instruction," (https://iris.peabody.vanderbilt.edu/module/dbi1/), and the Taxonomy of Interventions provided by the National Center on Intensive Intervention (https://intensiveintervention.org/taxonomy-intervention-intensity).

Summary

Academic performance assessments reveal that the majority of American students are not achieving benchmark expectations for mathematics (NCES, 2019). Response to Intervention is a comprehensive school improvement model designed to help all learners achieve academic proficiency. The core elements of RtI include: (1) providing high-quality instruction to prevent mathematics difficulties, (2) using data to guide instructional decision making and evaluate instructional effectiveness, and (3) providing support for students who are at risk of academic failure through multiple levels of increasingly intense, targeted interventions. In this chapter, we reviewed the components of high-quality instruction and described how to locate high-quality materials for use in each tier, as well as ideas for intensifying instruction when validated programs are not available. We discussed using data both to evaluate instructional effectiveness and to identify students at risk of academic failure, and we reviewed how RtI's tiered service delivery model can efficiently distribute instructional resources to provide early intervention for the greatest possible number of students. Using RtI effectively has the potential to allow all American students to become proficient in mathematics.

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